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About a construction and some analysis of time inhomogeneous diffusions on monotonely moving domains

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Abstract

We construct and analyze in a very general way time inhomogeneous (possibly also degenerate or reflected) diffusions in monotonely moving domains $E \subset \mathbb{R} \times \mathbb{R}^d$, i.e. if $E_t := \{x \in \mathbb{R}^d | (t,x) \in E\}$, $t \in \mathbb{R}$, then either $E_s \subset E_t$, $\forall s \leqslant t$, or $E_s \supset E_t$, $\forall s \leqslant t$, $s,t \in \mathbb{R}$. Our major tool is a further developed $L^2(E,m)$-analysis with well chosen reference measure m. Among few examples of completely different kinds, such as e.g. singular diffusions with reflection on moving Lipschitz domains in $\mathbb{R}^d$, non-conservative and exponential time scale diffusions, degenerate time inhomogeneous diffusions, we present an application to what we name skew Bessel process on γ. Here γ is either a monotonic function or a continuous Sobolev function. These diffusions form a natural generalization of the classical Bessel processes and skew Brownian motions, where the local time refers to the constant function $\gamma \equiv 0$.

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0. Introduction

The theory of time dependent Dirichlet forms has been inspired from [11,12], and recently by [13]. We present here an independent, self-contained and more enclosing finite dimensional analysis, being integrated in the general theory of generalized Dirichlet forms. The present exposition can also be seen as a completion to [17], and vice versa.

We are concerned with the development of a new technique for the general treatment of time dependent diffusions on monotonely moving domains $E \subset \mathbb{R} \times \mathbb{R}^d$ with generator

$$LF(s,x) = \sum_{i,j=1}^{d} a_{ij}(s,x)\partial_i\partial_j F(s,x) + \sum_{i=1}^{d} b_i(s,x)\partial_i F(s,x) - c(s,x)F(s,x) + d(s)\partial_t F(s,x),$$

where the (symmetric) diffusion matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant d}$ may be degenerate, b_i discontinuous, non locally bounded, c is positive and bounded, and L may be regarded together with some boundary conditions. At the present stage we give as examples $d \equiv 1$, and $d(s) = sd$, where d is a positive constant.

Actually L may be examined with absorbing boundary conditions on some "freely" chosen $J \subset \partial E$, and with reflecting boundary conditions on the complement $\partial E \setminus J$. In order to not overload the exposition of this paper we concentrate exclusively on reflecting boundary conditions in case of existent boundary (cf. Remark 1.10).

For $s \in \mathbb{R}$ let $E_s := \{x \in \mathbb{R}^d | (s,x) \in E\}$. A monotonely moving domain $E \subset \mathbb{R} \times \mathbb{R}^d$ is by our definition a closed domain which satisfies either $E_s \subset E_t$, or $E_t \subset E_s$, for all $s \leqslant t$ in some time interval of $\mathbb{R}$. If $E_s \equiv E_t$, we are concerned with "constantly moving domains". These are cylindrical domains such as e.g. $\mathbb{R} \times \mathbb{R}^d$, or more generally such as e.g. $\mathbb{R} \times D$, $D \subset \mathbb{R}^d$. For simplicity we assume that the time interval T_I is given by $\mathbb{R}$, or by $\mathbb{R}^+ := [0, \infty)$. We can write $E = \bigcup_{s \in T_I} \{s\} \times E_s$.

If

$$b_i = \frac{1}{2} \sum_{j=1}^{d} \left(\partial_j a_{ij} + a_{ij} \frac{\partial_j \rho}{\rho} \right)$$

with sufficiently regular ρ, $A = (a_{ij})_{1 \leqslant i,j \leqslant d}$, and ∂E, an integration by parts w.r.t. the "well-chosen" measure $dm = \rho(s,x)\,dx\,ds$ gives

$$-\int_E LF \cdot G\,dm = \frac{1}{2} \int_E \langle A\nabla F, \nabla G \rangle\,dm + \int_E cFG\,dm - \int_E \Lambda F \cdot G\,dm, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product in $\mathbb{R}^{d+1}$, and

$$\Lambda F(s,x) := d(s)\partial_t F(s,x).$$

The boundary conditions on L will make disappear the surface measure term on ∂E. In space dimension one, i.e. $d=1$, we can force the first order space derivative coefficient to be b, with very mild conditions on b and A, by letting $\rho(s,x) = \frac{1}{A(s,x)} e^{2\int_0^x \frac{b}{A}(s,x')dx'}$. In general, we observe that L is related to a bilinear form which has a symmetric part

$$\mathcal{A}(F,G) := \frac{1}{2}\int_E \langle A\nabla F, \nabla G\rangle\, dm + \int_E cFG\, dm$$

and a non-symmetric time derivative part

$$\mathcal{N}(F,G) := \int_E \Lambda F\cdot G\, dm.$$

Let $C_0^1(E)$ denote the one times continuously differentiable functions on $\mathbb{R}^{d+1}$ restricted to E. In contrary to the left-hand side of (1), $\mathcal{A}$, $\mathcal{N}$, can be defined for $F, G \in C_0^1(E)$, and of course without boundary conditions. Thus the right-hand side of (1) is a more general expression, which is better to use as a starting point.

The operator Λ occurring in $\mathcal{N}$, is given through a C^1-flow $\kappa : T_I \times \mathbb{R}^+ \to T_I$, i.e. $\kappa(s,0) = s$, $(\kappa(s,t), x) \in E$ whenever $(s,x)\in E$, $t\geqslant 0$, and

$$\forall s \in T_I, r, t \geqslant 0: \quad \kappa(s, r+t) = \kappa(\kappa(s,r), t)$$

The C_0-semigroup on $L^2(E,m)$ corresponding to the perturbation Λ of $\mathcal{A}$ is then determined through

$$U_t G(s,x) := G(\kappa(s,t), x) \text{ for every } (s,x)\in E, t\geqslant 0, G\in C_0^1(E),$$

so that

$$\lim_{t\to 0}\frac{1}{t}(U_t G - G) = \Lambda G \text{ in } L^2(E,m).$$

Hence

$$d(s) = \kappa'(s,0) := \lim_{t\to 0}\frac{\kappa(s,t)-\kappa(s,0)}{t}.$$

Our first observation is, that under the conditions on ρ, A, κ, which are more rigorously described in Section 1, there exists a regular generalized Dirichlet form $\mathcal{E}$ extending

$\mathcal{A}-\mathcal{N}$, and moreover a Hunt process Y_t with state space E associated to L (see Theorem 1.8). By Theorem 1.9 Y_t is a diffusion up to its life time ζ. The exceptional sets of Y_t are described through the strict (parabolic) capacity $\mathrm{Cap}_{1,\widehat{G}_1\Phi}$ defined at the end of Section 1. In practice, besides very weak assumptions on ρ, A, in order to obtain closability of $(\mathcal{A}(\cdot,\cdot), C_0^1(E))$ in $L^2(E,m)$ (cf. Lemma 1.1), the main regularity assumption will be a monotonicity assumption in the time variable of ρ, e.g.

$$\rho(s,\cdot)\leqslant\rho(t,\cdot)\,dx\text{-a.e. for any } s\leqslant t.$$

In fact, this is the main restriction of our method. The monotonicity assumption in time as well as on ρ than on E_s makes $(U_t)_{t\geqslant 0}$ (among others) a contraction semigroup on $L^2(E,m)$ and guarantees the negative definiteness of Λ, hence the positivity of $\mathcal{E}$. Indeed, $L-\Lambda$ is negative definite too, as it is related to the positive form $\mathcal{A}$.

Let us denote the closure of $(\mathcal{A}(\cdot,\cdot), C_0^1(E))$ in $L^2(E,m)$ by $(\mathcal{A},\mathcal{V})$. $(\mathcal{A},\mathcal{V})$ is a symmetric regular Dirichlet form on $L^2(E,m)$ (see [6]) and the Dirichlet space $\mathcal{V}$ is the closure of $C_0^1(E)$ w.r.t. $|F|_{\mathcal{V}} := \sqrt{\mathcal{A}(F,F)+\int_E |F|^2 dm}$. Let $\mathcal{V}'$ with norm $|\cdot|_{\mathcal{V}'}$ be the dual space of $\mathcal{V}$. Lemma 1.7 tells us that the domain of the generalized Dirichlet form $\mathcal{F}$ is the closure of $C_0^1(E)$ w.r.t. the norm given by

$$|F|_{\mathcal{F}} := \sqrt{|F|_{\mathcal{V}}^2 + |\Lambda F|_{\mathcal{V}'}^2}.$$

So, $C_0^1(E)$ is dense both in $\mathcal{V}$ and in $\mathcal{F}$.

By a general theorem (see [19, Theorem 4.5]) together with the refined potential theory of [22] we obtain the"extended" Fukushima decomposition for $F\in\mathcal{F}$ with strictly $\mathcal{E}$-quasi-continuous m-version $\widetilde{F}$, i.e.

$$A_t^{[f]} := \widetilde{F}(Y_t)-\widetilde{F}(Y_0) = M_t^{[F]} + N_t^{[F]};\ \ t\geqslant 0, \tag{2}$$

where $M^{[F]}$ is a continuous martingale additive functional of finite energy and $N^{[F]}$ is a continuous additive functional of zero energy (for the definitions see beginning of Section 2). Decomposition (2) is meant in the sense of equivalence of additive functionals of Y_t w.r.t. the strict (parabolic) capacity $\mathrm{Cap}_{1,\widehat{G}_1\Phi}$. The question under which conditions (2) becomes a point-wise equation will be the content of further investigation. It is related to the question whether the corresponding semigroup or resolvent to $\mathcal{E}$ is absolutely continuous w.r.t. the reference measure m.

In Section 2 we analyze more intensively the martingale and drift part of (2). By Lemma 2.1 the dual predictable projection $\langle M^{[F]}\rangle$, $F\in\mathcal{F}$, of the square bracket of $M^{[F]}$ has the following energy measure

$$\mu_{\langle M^{[F]}\rangle}(dy) = \langle A\nabla F,\nabla F\rangle\rho\,dy - cF^2\rho\,dy.$$

The right hand part is the so called killing measure. Theorem 2.2 tells us the following: Any positive Borel measure ν for which there exists a strict $\mathcal{E}$-nest $(\overline{E}_k)_{k\geqslant 1}$ such that

$1_{\overline{E}_k} dv$ is a finite smooth measure w.r.t. $\mathcal{A}$ can uniquely be determined by a PCAF of Y through the Revuz formula (11) and (18). This theorem is fundamental and an entirely new observation in time inhomogeneous theory. It gives us the possibility to construct and define even multi-dimensional local times on non smooth surfaces in a very general way. Theorem 2.3 provides then a standard method to identify the drift part in (2). Simple conservativity criteria are given at the end of Section 2 in Lemma 2.4.

In Section 3 we apply our just developed theory to four generic examples:

- Skew Bessel processes w.r.t. a continuous function of bounded variation.
- Degenerate time inhomogeneous diffusions.
- Non-conservative and exponential time scale diffusions.
- Singular time inhomogeneous diffusions with reflection on moving Lipschitz domains.

We will comment extensively only on the first, which is the most "simple" one. The other three might be directly looked at by the reader in Section 3. Of course the developed theory is so constructed that much more new examples of diffusions can be produced.

Let $\gamma : \mathbb{R}^+ \to \mathbb{R}$, be continuous and of bounded variation. Let $\eta \in \mathbb{R}$, $|\eta| \leqslant 1$. We call η-skew Bessel process of dimension $\delta \geqslant 1$ w.r.t. γ any solution (which by definition is a continuous semimartingale) to

$$X_t = X_0 + W_t + \frac{\delta - 1}{2} \int_0^t \frac{ds}{X_s} + \eta \widetilde{L}_t^0(X - \gamma), \tag{3}$$

where W is a standard Brownian motion starting from zero, and $\widetilde{L}^0(X - \gamma)$ is the symmetric local time at zero of the continuous semimartingale $X_t - \gamma(t)$, i.e.

$$\widetilde{L}_t^0(X - \gamma) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(X_s - \gamma(s))\, ds$$

(see [16, VI.(1.25) Exercise]). The terminology of η-skew Bessel process w.r.t. γ is justified because of the following: Assume $\gamma \equiv 0$. If $\delta = 1$ we clearly obtain the η-skew Brownian motion (you can find it also sometimes as $\frac{\eta+1}{2}$-skew Brownian motion in the literature). If on the other hand $\delta > 1$, we can compare $|X_t|$ (see Remark 3.3) obtained first through the symmetric Tanaka formula and second obtained as the square root of X_t^2 calculated by Itô's formula, in order to see that $\widetilde{L}_t^0(X) \equiv 0$. Thus in this case X_t is a Bessel process of dimension δ.

We arrive to construct a solution to (3) in different cases. We are mostly interested in the case $\delta \in [1, 2)$. If $\delta = 1$, and the distributional derivative γ' is in $L^2_{\mathrm{loc}}(dt)$, a solution can be derived by Girsanov transforming the classical η-skew Brownian motion on $\mathbb{R}$. After having finished this paper we remarked that the case $\eta = 1$ had already been done before us and independently in [2].

If $\delta = 1$, $\gamma' \in L^2_{\mathrm{loc}}(dt)$, $\eta \in [-1, 1]$, then strong uniqueness holds for (3). Furthermore, there does not exist a solution if $|\eta| > 1$. In fact, both is a direct consequence of the same results obtained for $\gamma \equiv 0$ in [7].

If $\delta \in (1,2)$, and γ is monotonely decreasing, with $\gamma' \in L^2_{\rm loc}(dt)$, then a solution to (3) is constructed using the underlying generalized Dirichlet form

$$\mathcal{E}(F,G)=\int_0^\infty \left(\frac{\alpha}{2}\int_{-\infty}^{\gamma(s)} \partial_x F\, \partial_x G\, |x|^{\delta-1}\, dx + \frac{\beta}{2}\int_{\gamma(s)}^{\infty} \partial_x F\, \partial_x G\, |x|^{\delta-1}\, dx\right) ds$$
$$- \int_E \partial_t F\, G\, \rho\, dy, \quad F, G \in C_0^1(E), \tag{4}$$

$\rho(s,x) := \left(\alpha 1_{(-\infty,\gamma(s))}(x) + \beta 1_{[\gamma(s),\infty)}(x)\right)|x|^{\delta-1}$, $0 \leqslant \alpha \leqslant \beta$, $\beta > 0$, $\eta = \frac{\beta-\alpha}{\beta+\alpha} \in [0,1]$. In this case $E = \{(s,x) \in \mathbb{R}^+ \times \mathbb{R} | x \geqslant \gamma(s)\}$ if $\alpha = 0$, $E = \mathbb{R}^+ \times \mathbb{R}$ otherwise. Note that $\rho(s,\cdot) \leqslant \rho(t,\cdot)$, that $E_s \subset E_t$, if $s \leqslant t$, $c \equiv 0$, and $\kappa(s,t) = s + t$.
If $\delta \in (1,2)$, and γ is monotonely increasing, with $\gamma' \in L^2_{\rm loc}(dt)$, similarly we would obtain a solution for any $\eta \in [-1,0]$.

Why do we have $\eta = \frac{\beta-\alpha}{\beta+\alpha}$? In order to answer to this question let us look at the more general case where $\delta \in [1,2)$, and γ is only monotonely decreasing. In this case, the diffusion of Theorem 1.9 corresponding to the generalized Dirichlet form (4) is

$$X_t = X_0 + W_t + \frac{\delta - 1}{2}\int_0^t \frac{ds}{X_s} + \frac{\beta - \alpha}{2}\, \ell_t^\gamma \tag{5}$$

and ℓ^γ is the unique positive continuous additive functional of (t, X_t) which is associated to the measure $|x|^{\delta-1}\delta_{\gamma(s)}(dx)\, ds$ via the Revuz formula (see Theorem 2.2, Lemma 3.2) with reference measure $m = \rho dy$. In fact, using integration by parts in (4) for $F(t,x) = x$ we can see that there occurs a term involving $\frac{\beta-\alpha}{2}|x|^{\delta-1}\delta_{\gamma(s)}(dx)\, ds$. Since the support of the measure $|x|^{\delta-1}\delta_{\gamma(s)}(dx)\, ds$ is located on γ the additive functional ℓ^γ increases only when $X_t = \gamma(t)$. This is clearly analogous to the classical case, i.e. $\delta, \beta = 1$, $\alpha = 0$, $\gamma \equiv 0$, where the local time in zero of the Brownian motion is described through the Dirac mass in zero (see e.g. [6] or [16]). Observe however that (5), as is (2), is in the sense of equivalence of additive functionals.

The reference measure corresponding to the generalized Dirichlet form (4) depends on α, β, and so does ℓ^γ. Assume $\gamma' \in L^2_{\rm loc}(dt)$. Applying $F(t,x) = |x - \gamma(t)|$ to the continuous semimartingale X_t in (5), we obtain (cf. Section 3)

$$|X_t - \gamma(t)| = |X_0 - \gamma(0)| + \int_0^t sign(X_s - \gamma(s))\, d(X_s - \gamma(s)) + \frac{\beta + \alpha}{2}\ell_t^\gamma.$$

Here $sign(x) := 1$ if $x > 0$, $sign(x) := 0$ if $x = 0$, $sign(x) := -1$ if $x < 0$, is the symmetric sign. Thus $\frac{\beta+\alpha}{2}\ell_t^\gamma$ must be the symmetric local time of the continuous semimartingale $X_t - \gamma(t)$ (see [16, VI.(1.25) Exercise]), and therefore $\eta = \frac{\beta-\alpha}{\beta+\alpha}$.

The diffusion corresponding to the generalized Dirichlet form (4) with $\delta \in (0,1)$ is constructed in Section 3 for γ monotonely decreasing but not yet identified. This will be done elsewhere.

On the other hand, if $\delta > 1$, a solution to (3) and (5), seems not to be in general attainable through a Girsanov transformation or by direct stochastic calculus. Moreover, we even do not have uniqueness of (3) in the putative simplest case $\gamma \equiv 0$ (see Remark 3.3).

Let us complete this example with an open question. Does there exist a solution to (3) for $|\eta| > 1$, $\delta \neq 1$? If $\delta = 1$ we have already said that there is no solution.

1. Definition of the generalized Dirichlet form and construction of the associated process

The notions used in this article if not explicitly defined here will be the usual ones of the previous articles [19–22]. If U is an open set in some Euclidean space let $C^k(U)$ (resp. $C_0^k(U)$) denote the k-times continuously differentiable functions (resp. with compact support) in U. On the other hand if $F \subset \mathbb{R}^n$ is closed let $C^k(F)$ (resp. $C_0^k(F)$) denote the restrictions to F of functions in $C^k(\mathbb{R}^n)$ (resp. $C_0^k(\mathbb{R}^n)$). For a function $f : C \to \mathbb{R}$ and $B \subset C$ let $f_{|B}$ denote the restriction of f on B. If $\mathcal{D}$ is a space of functions $f : C \to \mathbb{R}$ and $B \subset C$ we let $\mathcal{D}_{|B} := \{f_{|B} \,|\, f \in \mathcal{D}\}$. For an interval in $\mathbb{R}$ we use the symbols "[" or "]" in order to indicate that it is closed on one side and the symbols "(" or ")" in order to indicate that it is open on one side. We let $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^- = (-\infty, 0]$.

Let $d \geqslant 1$, and $dy = dx\,ds$ be the Lebesgue measure on $\mathbb{R}^{d+1}$, where dx is the Lebesgue measure on $\mathbb{R}^d$, ds the Lebesgue measure on $\mathbb{R}$. Let $E \subset \mathbb{R} \times \mathbb{R}^d$ be a closed space-time domain where each $E_t := \{x \in \mathbb{R}^d | (t, x) \in E\}$, $t \in \mathbb{R}$, is a closed, not necessarily bounded subset of the Euclidean space $\mathbb{R}^d$. We suppose $T_I := \mathbb{R} \setminus \{t \in \mathbb{R} | E_t = \emptyset\} = \mathbb{R}^+$, or $T_I = \mathbb{R}$, so that $E = \bigcup_{t \in T_I} \{t\} \times E_t$. We additionally assume that E_t moves monotonely, i.e. either $E_s \subset E_t$, $\forall s \leqslant t$, or $E_s \supset E_t$, $\forall s \leqslant t$, $s, t \in T_I$. Let finally the boundary ∂E of E have zero $(d+1)$-dimensional Lebesgue measure.

By $\mathcal{B}$, $\mathcal{B}^+$, $\mathcal{B}_b$, $\mathcal{B}_b^+$, we denote the Borel measurable, positive Borel measurable, bounded Borel measurable, bounded Borel measurable and positive functions on E. Let $A \subset \mathbb{R}^n$, $n \geqslant 1$, be Borel measurable. We denote by 1_A the characteristic function of A. For a Borel measure v on A, $p \in [1, \infty]$, we denote by $L^p_{\mathrm{loc}}(A, v)$ the space of all Borel measurable F on A for which $1_K F \in L^p(A, v)$ for any in compact subset K of A, and $L^p(A, v)$ denotes as usual the p-fold integrable functions on A w.r.t. v if $p < \infty$, and the v-essentially bounded functions on A if $p = \infty$. A function is called locally bounded, if it is bounded on compact sets. We denote by $|\cdot|$ the Euclidean norm, by $|\cdot|_\infty$ the sup norm in the corresponding space, i.e. either the point-wise sup norm or the essential sup norm w.r.t. a measure. Let $A \subset E$. We set $A^c := E \setminus A$, i.e. the complement of A in E.

Let $a_{ij} = a_{ji}, \rho : E \longrightarrow \mathbb{R}$, $1 \leqslant i, j \leqslant d$, be measurable functions with $\rho \in L^1_{\mathrm{loc}}(E, dy)$, $\rho > 0\, dy$-a.e. on E. We let $A = (a_{ij})_{1 \leqslant i,j \leqslant d}$ be the symmetric matrix given by the functions a_{ij}. For a function F we denote by $\partial_i F$ the ith partial distributional derivative in space direction, by ∇F the space gradient of F. Because $dy(\partial E) = 0$ we will identify $L^p(E, dy)$ and $L^p(E \setminus \partial E, dy)$. In connection with this we will consider a function $F : E \to \mathbb{R}$ as a function $F : E \setminus \partial E \to \mathbb{R}$ and vice versa. We assume that either for some constant $C \geqslant 1$

(A1) $C^{-1} \sum_{i=1}^{d} \xi_i^2 \leqslant \sum_{i,j=1}^{d} a_{ij}(y)\xi_i \xi_j \leqslant C \sum_{i=1}^{d} \xi_i^2;\ \forall(\xi_1, ..., \xi_d) \in \mathbb{R}^d, y \in E,$

or

(A1′) $E = T_I \times \mathbb{R}^d$, a_{ij} is locally bounded, $\partial_j a_{ij} \in L^2_{\mathrm{loc}}(E, \rho dy)$, $1 \leqslant i, j \leqslant d$, and

$$\sum_{i,j=1}^{d} a_{ij}(y)\xi_i \xi_j \geqslant 0 \quad \forall(\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, y \in E.$$

Let $\mathcal{H} := L^2(E, \rho\, dy)$ with norm $|F|_{\mathcal{H}} := (\int_E F(y)^2 \rho(y) dy)^{\frac{1}{2}}$, and inner product $(\cdot, \cdot)_{\mathcal{H}}$. For $\mathcal{G} \subset \mathcal{H}$, we set $\mathcal{G}_b := \mathcal{G} \cap \mathcal{B}_b$, $\mathcal{G}^+ := \mathcal{G} \cap \mathcal{B}^+$, $\mathcal{G}_b^+ := \mathcal{G} \cap \mathcal{B}_b^+$.

Let $c \in L^\infty(E, \rho\, dy)$, $c \geqslant 0$ ρdy-a.e., and consider the following bilinear form:

$$\begin{aligned}\mathcal{A}(F, G) &:= \frac{1}{2} \sum_{i,j=1}^{d} \int_{T_I} \int_{E_s} a_{ij}(s,x)\partial_i F(s,x)\partial_j G(s,x)\rho(s,x)\, dx\, ds \\ &\quad + \int_{T_I} \int_{E_s} c(s,x)F(s,x)G(s,x)\rho(s,x)\, dx\, ds; \qquad F, G \in C_0^1(E).\end{aligned}$$

Obviously $C_0^1(E) \subset \mathcal{H}$ dense, so $\mathcal{A}$ is densely defined on $\mathcal{H}$. We assume from now on that it is closable in $\mathcal{H}$, i.e. if $(F_n)_{n \in \mathbb{N}} \subset C_0^1(E)$ is an $\mathcal{A}$-Cauchy sequence converging to 0 in $\mathcal{H}$, then we must have $\lim_{n\to\infty} \mathcal{A}(F_n, F_n) = 0$. The closure $(\mathcal{A}, \mathcal{V})$ of $(\mathcal{A}, C_0^1(E))$ in $\mathcal{H}$ is a regular symmetric Dirichlet form on $\mathcal{H}$ (cf. [6]). Let $|F|_{\mathcal{V}}$ be the corresponding Dirichlet norm, i.e. $|F|_{\mathcal{V}} := (\mathcal{A}(F, F) + |F|^2_{\mathcal{H}})^{\frac{1}{2}}$, and $\mathcal{A}_\alpha(\cdot, \cdot) := \mathcal{A}(\cdot, \cdot) + \alpha(\cdot, \cdot)_{\mathcal{H}}$. Let $\mathrm{Cap}^{\mathcal{A}}$ be the capacity corresponding to $(\mathcal{A}, \mathcal{V})$ as a regular Dirichlet form, and let $K \subset E$ be a compact set. By [6, Lemma 2.2.7(ii)] we have

$$\mathrm{Cap}^{\mathcal{A}}(K) = \inf\{\mathcal{A}_1(F, F);\, F \in C^1_{0,K}(E)\} \tag{6}$$

where $C^1_{0,K}(E) = \{F \in C_0^1(E) | F(y) \geqslant 1, \forall y \in K\}$.

The following condition on ρ in Lemma 1.1(i) is called the Hamza type condition.

Lemma 1.1. (i) *Let* $(A1)$ *hold. Let*

$$\rho = 0\, dy\text{-}a.e. \text{ on } (E \setminus \partial E) \setminus R(\rho)$$

where

$$R(\rho) = \left\{ z \in E \setminus \partial E \,\middle|\, \int_{\{y \in E\setminus\partial E \,|\, |z-y| \leqslant \varepsilon\}} \frac{1}{\rho(y)}\, dy < \infty \text{ for some } \varepsilon > 0 \right\}.$$

Then $(\mathcal{A}, C_0^1(E))$ *is closable in* $\mathcal{H}$.

(ii) *Let (A1′) hold. Let* $\rho = \varphi^2$ *with* $\partial_i \varphi \in L^2_{\text{loc}}(T_I \times \mathbb{R}^d, dx\, ds)$, $1 \leqslant i \leqslant d$. *Then* $(\mathcal{A}, C_0^1(E))$ *is closable in* $\mathcal{H}$.

Proof. (i) Let $(F_n)_{n\in\mathbb{N}} \subset C_0^1(E)$ be $\mathcal{A}$-Cauchy, and $\lim_{n\to\infty} F_n = 0$ in $\mathcal{H}(= L^2(E, \rho dy))$. Since $(F_n)_{n\in\mathbb{N}}$ is $\mathcal{A}$-Cauchy, and (A1) holds, it follows that $(\partial_i F_n)_{n\in\mathbb{N}}$ is $\mathcal{H}$-Cauchy. Let hence $G_i := \lim_{n\to\infty} \partial_i F_n$ in $\mathcal{H}$, $1 \leqslant i \leqslant d$. We have to show $\lim_{n\to\infty} \mathcal{A}(F_n, F_n) = 0$. Let us remark that $R(\rho)$ is the largest open set $U \subset E \setminus \partial E$ such that $\rho^{-1} \in L^1_{\text{loc}}(U, dx)$. If $K \subset R(\rho)$ is compact, and $G \in \mathcal{H}$, then by the Cauchy–Schwarz inequality

$$\int_K |G|\, dy = \int_K (|G|\rho \cdot 1)\rho^{-1}\, dy \leqslant \left(\int_E G^2 \rho\, dy\right)^{\frac{1}{2}} \left(\int_K \rho^{-1}\, dy\right)^{\frac{1}{2}}.$$

Thus $\mathcal{H} \subset L^1_{\text{loc}}(R(\rho), dy)$ continuously, and therefore $\lim_{n\to\infty} F_n = 0$, $\lim_{n\to\infty} \partial_i F_n = G_i$ in $L^1_{\text{loc}}(R(\rho), dy)$, $1 \leqslant i \leqslant d$. Let $V \in C_0^\infty(R(\rho))$. It follows

$$0 = -\lim_{n\to\infty} \int_{R(\rho)} F_n \partial_i V\, dy = \lim_{n\to\infty} \int_{R(\rho)} \partial_i F_n V\, dy = \int_{R(\rho)} G_i V\, dy,$$

thus $G_i = 0$ dy-a.e. on $R(\rho)$. Since ρ is supposed to satisfy the Hamza type condition we obtain $G_i = 0$ ρdy-a.e. on $E \setminus \partial E$. Since $dy(E \setminus \partial E) = 0$, it follows $G_i = 0$ ρdy-a.e. on E. Obviously $\lim_{n\to\infty} (cF_n, F_n)_{\mathcal{H}} = 0$. Hence $\lim_{n\to\infty} \mathcal{A}(F_n, F_n) = 0$.

(ii) Let $F, G \in C_0^2(T_I \times \mathbb{R}^d)$. Then

$$\begin{aligned}
\mathcal{A}(F,G) &= \frac{1}{2} \sum_{i,j=1}^d \int_{T_I} \int_{\mathbb{R}^d} \{a_{ij} \partial_i F \partial_j G + cFG\} \rho\, dx\, ds + \int_{T_I} \int_{E_s} cFG\rho\, dx\, ds \\
&= \frac{1}{2} \sum_{i,j=1}^d \int_{T_I} \int_{\mathbb{R}^d} \big(\partial_j (a_{ij} \partial_i F G \rho) - \partial_j (a_{ij} \partial_i F \rho)\big) G\, dx\, ds \\
&\quad + \int_{T_I} \int_{E_s} cFG\rho\, dx\, ds \\
&= \int_{T_I} \int_{\mathbb{R}^d} -\frac{1}{2} \sum_{i,j=1}^d \left(a_{ij} \partial_i \partial_j F + \partial_j a_{ij} \partial_i F + a_{ij} \frac{\partial_j \rho}{\rho} \partial_i F \right) G\rho\, dx\, ds \\
&\quad + \int_{T_I} \int_{E_s} cFG\rho\, dx\, ds = (-LF, G)_{\mathcal{H}}.
\end{aligned}$$

Since $(L, C_0^2(T_I \times \mathbb{R}^d))$ is a symmetric operator on $\mathcal{H}$ the closability of $(\mathcal{A}, C_0^2(T_I \times \mathbb{R}^d))$ in $\mathcal{H}$ now follows by [8, I.Proposition 3.3]. Let $\chi \in C_0^\infty(\mathbb{R}^{d+1})^+$, $\int_{\mathbb{R}^{d+1}} \chi(y)$

$dy = 1$, $supp(\chi) \subset \{y \in \mathbb{R}^{d+1} | |y| \leqslant 1\}$, $\chi_\varepsilon(y) := \varepsilon^{-(d+1)}\chi(\frac{y}{\varepsilon})$, $F \in C_0^1(\mathbb{R}^{d+1})$. Then $F_\varepsilon(z) = \chi_\varepsilon \star F(z) := \int_{\mathbb{R}^{d+1}} \chi_\varepsilon(z-y)F(y)\,dy \in C_0^\infty(\mathbb{R}^{d+1})$, and $F_\varepsilon \to F$, $\partial_i F_\varepsilon \to \partial_i F$, $1 \leqslant i \leqslant d$, uniformly as $\varepsilon \downarrow 0$. Thus also $F_\varepsilon|_{T_I \times \mathbb{R}^d} \to F|_{T_I \times \mathbb{R}^d}$, $\partial_i F_\varepsilon|_{T_I \times \mathbb{R}^d} \to \partial_i F|_{T_I \times \mathbb{R}^d}$, $1 \leqslant i \leqslant d$, uniformly as $\varepsilon \downarrow 0$. Hence $C_0^1(T_I \times \mathbb{R}^d)$ is included in the closure of $C_0^2(T_I \times \mathbb{R}^d)$ w.r.t. the norm $|F|_{\mathcal{V}} = (\mathcal{A}(F,F) + (F,F)_{\mathcal{H}})^{\frac{1}{2}}$. Therefore $(\mathcal{A}, C_0^1(T_I \times \mathbb{R}^d))$ is also closable in $\mathcal{H}$. $\square$

At this point it will be convenient to recall some basic definitions of semigroups and generators. A family $(T_t)_{t \geqslant 0}$ of bounded linear operators from a Banach space $\mathcal{W}$ into $\mathcal{W}$ is called a *C_0-semigroup on $\mathcal{W}$* if $T_0 = id_{\mathcal{W}}$, $T_t T_s = T_{t+s}$ for all $t, s \geqslant 0$, and $\lim_{t \to 0} T_t F = F$ in $\mathcal{W}$ for all $F \in \mathcal{W}$. It is called a *C_0-semigroup of contractions on $\mathcal{W}$* if additionally $|T_t F|_{\mathcal{W}} \leqslant |F|_{\mathcal{W}}$ for any $F \in \mathcal{W}$, $t \geqslant 0$.

Let $(T_t)_{t \geqslant 0}$ be a C_0-semigroup on $\mathcal{W}$. The linear operator $\mathcal{L}F := \lim_{t \to 0} \frac{1}{t}(T_t F - F)$ with domain

$$D(\mathcal{L}, \mathcal{W}) = \left\{ F \in \mathcal{W} \,|\, \exists \lim_{t \to 0} \frac{1}{t}(T_t F - F) \text{ in } \mathcal{W} \right\}$$

is called the (*infinitesimal*) *generator* of $(T_t)_{t \geqslant 0}$.

A bounded linear operator $U : \mathcal{W} \to \mathcal{W}$ is called *sub-Markovian* if $0 \leqslant UF \leqslant 1$ whenever $0 \leqslant F \leqslant 1$, $F \in \mathcal{W}$. A C_0-semigroup $(T_t)_{t \geqslant 0}$ on $\mathcal{H}$ is called *sub-Markovian* if T_t is sub-Markovian for all $t \geqslant 0$.

Let $\kappa : T_I \times \mathbb{R}^+ \to T_I$, $\kappa \in C^1(T_I \times \mathbb{R}^+)$, $\kappa(s, 0) = s$. We assume $(\kappa(s,t), x) \in E$ whenever $y = (s, x) \in E$, $t \geqslant 0$, and

(A2) $\forall s \in T_I, r, t \geqslant 0 : \ \kappa(s, r+t) = \kappa(\kappa(s, r), t)$.

We will now define the C_0-semigroup corresponding to the perturbation of $\mathcal{A}$. For $G \in C_0^1(E)$, $t \geqslant 0$, let first

$$U_t G(y) := G(\kappa(s,t), x) \text{ for every } y = (s, x) \in E, t \geqslant 0.$$

(A2) implies the semigroup property of $(U_t)_{t \geqslant 0}$. In order to have $U_t C_0^1(E) \subset C_0^1(E)$, $t \geqslant 0$, we assume that $G(\kappa(s,t), x) \in C_0^1(E)$ whenever $G \in C_0^1(E)$, $t \geqslant 0$. Let $G \in C_0^1(E)$. We suppose for any $T > 0$ the existence of a compact set K such that the support of $U_t G - G$, $t \in [0, T]$, is located in K. We assume further

(A3) $\int_E G(\kappa(s,t), x)\rho(s, x)\,dx\,ds \leqslant \int_E G\rho\,dy \qquad \forall G \in C_0^1(E)^+$, $t \geqslant 0$.

and the existence of constants $M \geqslant 1$, $\omega \geqslant 0$, such that

(A4) $\mathcal{A}(U_t G, U_t G) \leqslant M e^{\omega t} \mathcal{A}(G, G)$ $\forall G \in C_0^1(E)$, $t \geqslant 0$.

Note that (A3), resp. (A4), is in particular a condition on ρ, resp. c. We give concrete examples for $(a_{ij})_{1 \leqslant i,j \leqslant d}$, E, ρ, c, κ satisfying (A1), (A2), (A3), (A4), or (A1′), (A2), (A3), (A4), in the section “Examples” (see also statement of Lemma 2.4, Remark 1.4).

Using (A3) we obtain $|U_t G|_{\mathcal{H}} \leqslant |G|_{\mathcal{H}}$ for any $G \in C_0^1(E)$. Since $C_0^1(E) \subset \mathcal{H}$ dense, U_t induces a (linear) contraction on $\mathcal{H}$ which we also denote by U_t. For $(s,u) \in T_I \times \mathbb{R}^+$ let

$$\kappa'(s,u) := \lim_{t\to 0} \frac{\kappa(s,u+t) - \kappa(s,u)}{t},$$

then:

Lemma 1.2. $(U_t)_{t\geqslant 0}$ *is a sub-Markovian* C_0*-semigroup of contractions on* $\mathcal{H}$. *The corresponding generator* $(\Lambda, D(\Lambda,\mathcal{H}))$ *is an extension of* $\Lambda F = \kappa'(\cdot,0)\partial_t F$, $F \in C_0^1(E)$.

Proof. We have $0 \leqslant U_t G(y) = G(\kappa(s,t),x) \leqslant 1$, $y = (s,x) \in E$, for any $G \in C_0^1(E)$ with $0 \leqslant G \leqslant 1$. This obviously extends by density to $G \in \mathcal{H}$ with $0 \leqslant G \leqslant 1$. Hence $(U_t)_{t\geqslant 0}$ is sub-Markovian. Let $F \in C_0^1(E)$, $T > 0$. By assumption the support of $U_t F - F$, $t \in [0,T]$, is located in some common compact set K. Let $S := \sup\{|s|\,|(s,x) \in K\}$. Since $s = \kappa(s,0)$, we have for any $y = (s,x) \in E$, $t \in [0,T]$

$$\begin{aligned} |U_t F(y) - F(y)| &\leqslant |\,(F(\kappa(s,t),x) - F(s,x))\,| \\ &\leqslant |\partial_t F|_\infty |\kappa(s,t) - \kappa(s,0)| \\ &\leqslant |\partial_t F|_\infty \sup_{(s,u)\in[-S,S]\times[0,T]} |\kappa'(s,u)||t|. \end{aligned} \tag{7}$$

Hence $|U_t F(y) - F(y)|$ is uniformly bounded in y and in $t \in [0,T]$. Since $\lim_{t\downarrow 0} U_t F(y) = F(y)$ point-wise, we obtain $\lim_{t\downarrow 0} U_t F = F$ in $\mathcal{H}$ using Lebesgue's theorem. Since U_t, $t \geqslant 0$, is a contraction on $\mathcal{H}$, and $C_0^1(E) \subset \mathcal{H}$ dense, the strong continuity of $(U_t)_{t\geqslant 0}$ on $\mathcal{H}$ easily follows. On the other hand, if $0 \leqslant r,t$, $y = (s,x) \in E$, then by (A2)

$$U_r U_t F(y) = U_t(F(\kappa(s,r),x)) = F(\kappa(\kappa(s,r),t),x) = F(\kappa(s,r+t),x) = U_{r+t}F(y).$$

Since U_t, $t \geqslant 0$, is a contraction on $\mathcal{H}$, and $C_0^1(E) \subset \mathcal{H}$ dense, the last obviously also extends to $\mathcal{H}$. Therefore $(U_t)_{t\geqslant 0}$ is a sub-Markovian C_0-semigroup of contractions on $\mathcal{H}$. Finally, if $y = (s,x) \in E$, then $\kappa(y,0) = s$, hence

$$\frac{U_t F(y) - F(y)}{t} = \frac{F(\kappa(s,t),x) - F(\kappa(s,0),x)}{t}$$

converges point-wise to $\kappa'(s,0)\partial_t F(y)$. Since $|\frac{U_t F - F}{t}|$ is uniformly bounded in t by (7) and supported by K if $t \in [0,T]$ we obtain also convergence in $\mathcal{H}$ by Lebesgue's theorem. $\square$

Furthermore:

Lemma 1.3. *$(U_t)_{t\geqslant 0}$ can be restricted to a C_0-semigroup on $\mathcal{V}$.*

Proof. By (A3), (A4), for any $G \in C_0^1(E)$

$$|U_t G|_{\mathcal{V}}^2 = |U_t G|_{\mathcal{H}}^2 + \mathcal{A}(U_t G, U_t G) \leqslant M e^{\omega t} |G|_{\mathcal{V}}^2.$$

The assertion now follows in particular from [18, I.2 Lemma 2.2.]. $\square$

We denote by $(\Lambda, D(\Lambda, \mathcal{V}))$ the generator corresponding to the restriction of $(U_t)_{t\geqslant 0}$ on $\mathcal{V}$, and let $(\widehat{U}_t)_{t\geqslant 0}$ be the adjoint semigroup of $(U_t)_{t\geqslant 0}$ in $\mathcal{H}$, i.e. $\widehat{U}_t$ is the adjoint operator (on $\mathcal{H}$) of U_t for every $t\geqslant 0$.

Remark 1.4. In general it is not possible to extend $(U_t)_{t\geqslant 0}$ to a C_0-semigroup on the dual space $\mathcal{V}'$ of $\mathcal{V}$. Indeed, this would be equivalent to the possibility of restricting $(\widehat{U}_t)_{t\geqslant 0}$ to a C_0-semigroup on $\mathcal{V}$. But we will give an example where $(\widehat{U}_t)_{t\geqslant 0}$ cannot be restricted to a C_0-semigroup on $\mathcal{V}$: Let $E = \mathbb{R}\times[-1,1]$, $a_{ij} = \delta_{ij}$, $t\geqslant 0$. Let $c \equiv 1$, $\kappa(s,t) = s+t$, $\forall (s,t) \in \mathbb{R}\times\mathbb{R}^+$. Then a.e.

$$\widehat{U}_t G(s,x) = \frac{\rho(s-t,x)}{\rho(s,x)}\, G(s-t,x); \;\; F \in C_0^1(E).$$

Let us choose $\rho(s,x) = |x|^{\alpha} 1_{(\gamma(s),1]}(x) + |x|^{\beta} 1_{[-1,\gamma(s)]}(x)$, $0 < \beta < \alpha$, $\gamma(s) := arctan(s)$. Then a.e.

$$\widehat{U}_t G(s,x) = \left(1_{(\gamma(s),1]}(x) + 1_{(\gamma(s-t),\gamma(s)]}(x)|x|^{\alpha-\beta} + 1_{[-1,\gamma(s-t)]}(x)\right) G(s-t,x), \;\; t>0.$$

If e.g. $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$, and $G \in C_0^1(E)$, $G = 1$ on $[-1,1]\times[-1,1]$, then the squared space derivative of $\widehat{U}_t G$ multiplied with ρ is of order $|x|^{-5/4}$, hence not integrable in $\{(s,x)|\gamma(s-t) < x \leqslant \gamma(s)\}$. Thus there exists $G \in \mathcal{V}$ with $\widehat{U}_t G \notin \mathcal{V}$, $t>0$, and $(\widehat{U}_t)_{t\geqslant 0}$ can even not be restricted to $\mathcal{V}$.

Although in general $(\widehat{U}_t)_{t\geqslant 0}$ cannot be restricted to a C_0-semigroup on $\mathcal{V}$ we have:

Lemma 1.5. *$(\widehat{U}_t)_{t\geqslant 0}$ is sub-Markovian.*

Proof. Let $F \in \mathcal{H}$, $0\leqslant F\leqslant 1$, $t\geqslant 0$. Of course $0\leqslant \widehat{U}_t F$. On the other hand for any $V, G \in C_0^1(E)^+$, we have using (A3)

$$\begin{aligned}(V - V\,\widehat{U}_t F, G)_{\mathcal{H}} &= (V,G)_{\mathcal{H}} - (F, U_t(VG))_{\mathcal{H}} \\ &\geqslant \int_E \{VG(s,x) - VG(\kappa(s,t),x)\}\rho(s,x)\,dx\,ds \geqslant 0.\end{aligned}$$

This implies $V \geqslant V\,\widehat{U}_t F$. Thus, choosing $V_n \in C_0^1(E)$, $0 \leqslant V_n \uparrow 1$, we obtain $\widehat{U}_t F \leqslant 1$. $\square$

It follows from Lemmas 1.2, 1.3, that $(\Lambda, D(\Lambda, \mathcal{H}))$ is the generator of a C_0-semigroup of contractions on $\mathcal{H}$ that can be restricted to a C_0-semigroup on $\mathcal{V}$. Hence [18, I.Lemma 2.3.] implies that

$$\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}'$$

is closable as an operator from $\mathcal{V}$ to $\mathcal{V}'$. Let $(\Lambda, \mathcal{F})$ denote its closure. For the following up to the definition of the generalized Dirichlet form see [18]. $\mathcal{F}$ is a real Hilbert space with norm

$$|F|_{\mathcal{F}} := \left(|F|_{\mathcal{V}}^2 + |\Lambda F|_{\mathcal{V}'}^2\right)^{\frac{1}{2}}.$$

$(\widehat{U}_t)_{t \geqslant 0}$ can be extended to a C_0-semigroup on $\mathcal{V}'$. The corresponding generator $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{V}'))$ is the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$. $\widehat{\mathcal{F}} := D(\widehat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$ is a real Hilbert space with norm

$$|F|_{\widehat{\mathcal{F}}} := \left(|F|_{\mathcal{V}}^2 + |\widehat{\Lambda} F|_{\mathcal{V}'}^2\right)^{\frac{1}{2}}.$$

Let $\langle \cdot, \cdot \rangle$ be the dualization between $\mathcal{V}'$ and $\mathcal{V}$. The *generalized Dirichlet form* is now given through

$$\mathcal{E}(F, G) := \begin{cases} \mathcal{A}(F, G) - \langle \Lambda F, G \rangle & \text{for } F \in \mathcal{F},\ \ G \in \mathcal{V} \\ \mathcal{A}(F, G) - \langle \widehat{\Lambda} G, F \rangle & \text{for } G \in \widehat{\mathcal{F}},\ \ F \in \mathcal{V}. \end{cases}$$

Note that $\langle \cdot, \cdot \rangle$ when restricted to $\mathcal{H} \times \mathcal{V}$ coincides with $(\cdot, \cdot)_{\mathcal{H}}$. Let $\mathcal{E}_\alpha(F, G) := \mathcal{E}(F, G) + \alpha(F, G)_{\mathcal{H}}$ for $\alpha > 0$. It follows, from [18, I.Proposition 3.4, p. 19], that for all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \to \mathcal{F}$ and $\widehat{W}_\alpha : \mathcal{V}' \to \widehat{\mathcal{F}}$ such that $\mathcal{E}_\alpha(W_\alpha F, G) = \langle F, G \rangle = \mathcal{E}_\alpha(G, \widehat{W}_\alpha F)$, $\forall F \in \mathcal{V}'$, $G \in \mathcal{V}$. Furthermore $(W_\alpha)_{\alpha > 0}$ and $(\widehat{W}_\alpha)_{\alpha > 0}$ satisfy the resolvent equation

$$W_\alpha - W_\beta = (\beta - \alpha) W_\alpha W_\beta \quad \text{and} \quad \widehat{W}_\alpha - \widehat{W}_\beta = (\beta - \alpha) \widehat{W}_\alpha \widehat{W}_\beta.$$

Restricting W_α to $\mathcal{H}$ we get a strongly continuous contraction resolvent $(G_\alpha)_{\alpha > 0}$ on $\mathcal{H}$ satisfying $\lim_{\alpha \to \infty} \alpha G_\alpha F = F$ in $\mathcal{V}$ for all $F \in \mathcal{V}$. The resolvent $(G_\alpha)_{\alpha > 0}$ is called the *resolvent associated with* $\mathcal{E}$. Let $(\widehat{G}_\alpha)_{\alpha > 0}$ be the adjoint of $(G_\alpha)_{\alpha > 0}$ in $\mathcal{H}$. $(\widehat{G}_\alpha)_{\alpha > 0}$ is called the *coresolvent associated with* $\mathcal{E}$.

We define an intermediate space $\mathcal{V}^{\mathcal{F}} := \{H \in \mathcal{H} \mid \sup_{\alpha>0} \alpha(H - \alpha G_\alpha H, H)_{\mathcal{H}} < \infty\}$. It is known, that $\mathcal{F} \cup \widehat{\mathcal{F}} \subset \mathcal{V}^{\mathcal{F}} \subset \mathcal{V}$, and that normal contractions operate on $\mathcal{V}^{\mathcal{F}}$ (see e.g. [20]). Furthermore if $F \in \mathcal{V}^{\mathcal{F}}$, then $\alpha\widehat{G}_\alpha F$ converges weakly to F in $\mathcal{V}$ as $\alpha \to \infty$. This follows immediately from the Banach-Alaoglu theorem and the inequalities

$$\mathcal{A}_1(\alpha\widehat{G}_\alpha F, \alpha\widehat{G}_\alpha F) \leqslant \mathcal{E}_1(\alpha\widehat{G}_\alpha F, \alpha\widehat{G}_\alpha F) \leqslant \mathcal{E}_1(F, \alpha\widehat{G}_\alpha F)$$
$$\leqslant |F|_{\mathcal{H}} + \sup_{\alpha>0} \alpha(F - \alpha G_\alpha F, F)_{\mathcal{H}}.$$

An element u of $\mathcal{H}$ is called 1-*excessive* (resp. 1-*coexcessive*) if $\beta G_{\beta+1} u \leqslant u$ (resp. $\beta\widehat{G}_{\beta+1} u \leqslant u$) for all $\beta \geqslant 0$. Let $\mathcal{P}$ (resp. $\widehat{\mathcal{P}}$) denote the 1-excessive (resp. 1-coexcessive) elements of $\mathcal{V}$. Let $\mathcal{C},\mathcal{D} \subset \mathcal{H}$. We define $\mathcal{D}_{\mathcal{C}} := \{u \in \mathcal{D} \mid \exists f \in \mathcal{C},\ u \leqslant f\}$. For an arbitrary Borel set B and an element $u \in \mathcal{H}$ such that $\{v \in \mathcal{H} \mid v \geqslant u \cdot 1_B\} \cap \mathcal{F} \neq \emptyset$ (resp. $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$) let $u_B := e_{u \cdot 1_B}$ be the 1-*reduced function* (resp. $\hat{u}_B := \hat{e}_{\hat{u} \cdot 1_B}$ be the 1-*coreduced function*) of $u \cdot 1_B$ (resp. $\hat{u} \cdot 1_B$) as defined in [18, Definition III.1.8, p. 65]. Note that in general only if B is open our definition of reduced function coincides with the one of [6, p. 92], [8, Exercise III.3.10(ii), p. 84]. In particular, if B is a Borel set such that $\rho dy(B) = 0$, then $u_B = 0$. If $B = E$ we rather use the notation e_u instead of u_E.

$(G_\alpha)_{\alpha>0}$ (resp. $(\widehat{G}_\alpha)_{\alpha>0}$) is called *sub-Markovian*, if αG_α (resp. $\alpha\widehat{G}_\alpha$) is sub-Markovian for any α, i.e. $0 \leqslant \alpha G_\alpha F \leqslant 1$ (resp. $0 \leqslant \alpha\widehat{G}_\alpha F \leqslant 1$) whenever $0 \leqslant F \leqslant 1$, $F \in \mathcal{H}$, $\alpha > 0$.

Lemma 1.6. *The resolvent* $(G_\alpha)_{\alpha>0}$ *and the coresolvent* $(\widehat{G}_\alpha)_{\alpha>0}$ *associated with* $\mathcal{E}$ *are sub-Markovian.*

Proof. $(U_t)_{t\geqslant 0}$ is sub-Markovian. Hence, by Ma and Röckner [8, I.Proposition 4.3] $(\Lambda, D(\Lambda, \mathcal{H}))$ is a Dirichlet operator. Since additionally $(\mathcal{A}, \mathcal{V})$ is a Dirichlet form it follows by [18, I.Proposition 4.7] that $(G_\alpha)_{\alpha>0}$ is sub-Markovian. Since $(\widehat{U}_t)_{t\geqslant 0}$ is sub-Markovian by Lemma 1.5 it follows analogously that $(\widehat{G}_\alpha)_{\alpha>0}$ is sub-Markovian. □

Lemma 1.7. *We have* $C_0^1(E) \subset \mathcal{F}$ *dense.*

Proof. Since $D(\Lambda, \mathcal{V}) \subset \mathcal{F}$ dense by Stannat [18, I.2 Lemma 2.5.] it is enough to show that $C_0^1(E) \subset D(\Lambda, \mathcal{V})$ dense w.r.t. the graph norm. Since $(\Lambda, D(\Lambda, \mathcal{V}))$ is by Lemma 1.3 the generator of the C_0-semigroup $(U_t)_{t\geqslant 0}$ on $\mathcal{V}$, $U_t F \in C_0^1(E)$, for any $F \in C_0^1(E)$, $t \geqslant 0$, i.e. $C_0^1(E)$ is invariant under $(U_t)_{t\geqslant 0}$, and $C_0^1(E) \subset \mathcal{V}$ dense, we can apply [4, 1.7 Proposition] in order to obtain that $C_0^1(E) \subset D(\Lambda, \mathcal{V})$ dense w.r.t. the graph norm. □

Let us define the strict capacity corresponding to $\mathcal{E}$. We fix $\Phi \in L^1(E, \rho dy)$, $0 < \Phi \leqslant 1$, and let

$$\mathrm{Cap}_{1,\widehat{G}_1\Phi}(U) = \lim_{k\to\infty} \int_E (kG_1\Phi \wedge 1)_U \Phi\rho\, dy \text{ if } U \subset E \text{ is open,}$$

and

$$\mathrm{Cap}_{1,\widehat{G}_1\Phi}(A) = \inf\{\mathrm{Cap}_{1,\widehat{G}_1\Phi}(U) | U \supset A, U \text{ open}\} \quad \text{if } A \subset E \text{ arbitrary.}$$

We adjoin an extra point Δ to E and let $E_\Delta := E \cup \{\Delta\}$ be the one point compactification of E. As usual any function defined on E is extended to E_Δ putting $f(\Delta) = 0$. Given an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E, we define

$$C_\infty(\{F_k\}) = \left\{ f : A \to \mathbb{R} \,\middle|\, \bigcup_{k\geqslant 1} F_k \subset A \subset E,\ f_{|F_k \cup \{\Delta\}} \text{ is continuous } \forall k \right\}.$$

A subset $N \subset E$ is called strictly $\mathcal{E}$-exceptional if $\mathrm{Cap}_{1,\widehat{G}_1\Phi}(N) = 0$. An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called a strict $\mathcal{E}$-nest if $\mathrm{Cap}_{1,\widehat{G}_1\Phi}(F_k^c) \downarrow 0$ as $k \to \infty$. A property of points in E holds strictly $\mathcal{E}$-quasi-everywhere (s.$\mathcal{E}$-q.e.) if the property holds outside some strictly $\mathcal{E}$-exceptional set. A function f defined up to some strictly $\mathcal{E}$-exceptional set $N \subset E$ is called strictly $\mathcal{E}$-quasi-continuous (s.$\mathcal{E}$-q.c.) if there exists a strict $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$, such that $f \in C_\infty(\{F_k\})$.

Since $(\mathcal{E}, \mathcal{F})$ is regular, i.e. $C_0(E) \cap \mathcal{F}$ is dense in $C_0(E)$ w.r.t. the uniform norm as well as in $\mathcal{F}$, it follows that $(\mathcal{E}, \mathcal{F})$ is a strictly quasi-regular generalized Dirichlet form on E (see [22, Proposition 0.4]). On the other hand we can see from Lemma 1.7 that $\mathcal{G} := C_0^1(E)$ satisfies the condition SD3 of [22]. Hence, the following theorem is an immediate consequence of [22, Theorem 0.13].

Theorem 1.8. *There exists a Hunt process $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t\geqslant 0}, (Y_t)_{t\geqslant 0}, (P_y)_{y\in E_\Delta})$ with state space E, life time ζ, such that $R_\alpha F(y) := \int_0^\infty \int_\Omega e^{-\alpha t} F(Y_t(\omega))\, dP_y\, dt$ is a s.$\mathcal{E}$-q.c. ρdy-version of $G_\alpha F$ for any $\alpha > 0$ and any $F \in \mathcal{H}_b$.*

For a subset $A \subset E_\Delta$ let $\sigma_A := \inf\{t > 0 \mid Y_t \in A\}$ (resp. $D_A = \inf\{t \geqslant 0 | Y_t \in A\}$) be the *first hitting time* (resp. *first entry time*) w.r.t. $\mathbb{M}$. For a Borel measure ν on E and a Borel set B let $P_\nu(B) := \int_E P_y(B)\nu(dy)$ and E_ν be the expectation w.r.t. P_ν. As usual we denote by E_y the expectation w.r.t. P_y. If $U \subset E$ is open, then

$$\mathrm{Cap}_{1,\widehat{G}_1\Phi}(U) = \int_E E_y[e^{-\sigma_U}]\Phi(y)\rho(y)\, dy. \tag{8}$$

If $B \subset E$ is an arbitrary Borel measurable set, then

$$\mathrm{Cap}_{1,\widehat{G}_1\Phi}(B) = \int_E E_y[e^{-D_B}]\Phi(y)\rho(y)\,dy.$$

Both follows from [22, Lemma 0.8].

By strict quasi-regularity every element in $\mathcal{F}$ admits a strictly $\mathcal{E}$-q.c. ρdy-version (see [22, Proposition 0.9]). For a subset $\mathcal{D} \subset \mathcal{H}$ denote by $\widetilde{\mathcal{D}}^{\mathrm{str}}$ all the s.$\mathcal{E}$-q.c. ρdy-versions of elements in $\mathcal{D}$. In particular $\widetilde{\mathcal{P}}_{\mathcal{F}}^{\mathrm{str}}$ denotes the set of all s.$\mathcal{E}$-q.c. ρdy-versions of 1-excessive elements in $\mathcal{V}$ which are dominated by elements of $\mathcal{F}$. We have an analogy, namely [22, Theorem 0.16], to [19, Theorem 2.3]. That is: Let $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$. Then there exists a unique σ-finite and positive measure $\mu_{\hat{u}}^{\mathrm{str}}$ on $(E, \mathcal{B}(E))$ charging no strictly $\mathcal{E}$-exceptional set, such that

$$\int_E \widetilde{f}\, d\mu_{\hat{u}}^{\mathrm{str}} = \lim_{\alpha\to\infty} \mathcal{E}_1(f, \alpha\widehat{G}_{\alpha+1}\hat{u}) \quad \forall \widetilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}}^{\mathrm{str}} - \widetilde{\mathcal{P}}_{\mathcal{F}}^{\mathrm{str}}.$$

Also in analogy to [19] we introduce the following class of measures

$$\widehat{S}_{00}^{\mathrm{str}} := \{\mu_{\hat{u}}^{\mathrm{str}} \mid \hat{u} \in \widehat{\mathcal{P}}_{\widehat{G}_1\mathcal{H}_b^+} \text{ and } \mu_{\hat{u}}^{\mathrm{str}}(E) < \infty\},$$

where $\widehat{G}_1\mathcal{H}_b^+ := \{\widehat{G}_1 h \mid h \in \mathcal{H}_b^+\}$.

For $B \in \mathcal{B}(E)$ the following is known from [22, Theorem 0.17]: B is strictly $\mathcal{E}$-exceptional if, and only if $\mu(B) = 0$ for all μ in $\widehat{S}_{00}^{\mathrm{str}}$.

The proof of the following theorem is similar to [21, Theorem 2.5], or [20, Theorem 3.3]. We therefore omit it. Note however that due to the just described refined potential theory (see also [22, Remark 0.17]) we obtain statements w.r.t. the strict capacity defined above.

Theorem 1.9. *There exists a strictly $\mathcal{E}$-exceptional set $N \subset E$ such that*

$$P_y\big(t \mapsto Y_t \text{ is continuous on } [0,\zeta)\big) = 1 \text{ for every } y \in E \setminus N.$$

Remark 1.10. Let $E \setminus \partial E \subset E' \subset E$, $E' \subset \mathbb{R}^{d+1}$ measurable, and $\mathcal{Y} \subset C_0^1(E)$ be a subalgebra such that $\mathcal{Y} \subset C_0(E')$, and $\mathcal{Y}$ is dense in $C_0(E')$ w.r.t. the uniform norm. We assumed $\mathcal{A}(F,G)$; $F, G \in C_0^1(E)$, to be closable in $\mathcal{H}$. Since ∂E has zero $(d+1)$-dimensional Lebesgue measure, $(\mathcal{A}, \mathcal{Y})$ is clearly also closable in $L^2(E', m) \cong \mathcal{H}$. We denote the closure by $(\mathcal{A}, \mathcal{V}')$. Then $(\mathcal{A}, \mathcal{V}')$ is regular on E'. In order to have a representation for the capacity through the form $\mathcal{A}$ as in (6) we suppose that $\mathcal{Y}$ is a special standard core for $(\mathcal{A}, \mathcal{V}')$ (see [6, p. 6] for the definition of special standard core). We may then modify in an obvious way our assumptions (A3), (A4), together with some other small modifications and obtain the statements corresponding to Lemmas 1.2, 1.3, 1.5, 1.6, 1.7, Theorem 1.8, 1.9, with $(\mathcal{A}, \mathcal{V})$ replaced by $(\mathcal{A}, \mathcal{V}')$.

In this way it is for instance possible to construct diffusions with absorbing boundary conditions on $J \subset \partial E$, and with reflecting boundary conditions on the complement $\partial E \setminus J$. In order to not overload the present exposition we consciously abandoned the last.

2. Properties and some analysis of the associated process

Let us first recall some basic definitions and recent facts about additive functionals related to generalized Dirichlet forms.

A family $(A_t)_{t\geqslant 0}$ of extended real valued functions on Ω is called an *additive functional* (abbreviated AF) of $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t\geqslant 0}, (Y_t)_{t\geqslant 0}, (P_y)_{y\in E_\Delta})$ (w.r.t. $\mathrm{Cap}_{1,\widehat{G}_1\Phi}$), if:

(i) $A_t(\cdot)$ is $\mathcal{F}_t$-measurable for all $t\geqslant 0$.

(ii) There exists a *defining* set $\Lambda \in \mathcal{F}_\infty$ and a strictly $\mathcal{E}$-exceptional set $N \subset E$, such that $P_y(\Lambda) = 1$ for all $y \in E \setminus N$, $\theta_t(\Lambda) \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right continuous on $[0,\infty)$ and has left limits on $(0,\zeta(\omega))$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geqslant \zeta(\omega)$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geqslant 0$.

An AF A is called a *continuous additive functional* (abbreviated CAF), if $t \mapsto A_t(\omega)$ is continuous on $[0,\infty)$, a *positive, continuous additive functional* (abbreviated PCAF) if $A_t(\omega)\geqslant 0$ and a finite AF, if $| A_t(\omega) |< \infty$ for all $t\geqslant 0, \omega \in \Lambda$. Two AF's A,B are said to be equivalent (in notation $A = B$) if for each $t > 0$ $P_y(A_t = B_t) = 1$ for strictly $\mathcal{E}$-q.e. $y \in E$. The *energy* of an AF A of $\mathbb{M}$ is defined by

$$e(A) = \lim_{\alpha\to\infty} \frac{1}{2}\,\alpha^2 E_{\rho\,dy}\left[\int_0^\infty e^{-\alpha t} A_t^2\, dt\right], \tag{9}$$

whenever this limit exists in $[0,\infty]$. We will set $\overline{e}(A)$ for the same expression but with $\overline{\lim}$ instead of lim.

Let $\widetilde{F}$ be a strictly $\mathcal{E}$-q.c. ρdy-version of some element in $\mathcal{H}$. The additive functional

$$A^{[F]} := (\widetilde{F}(Y_t) - \widetilde{F}(Y_0))_{t\geqslant 0}$$

is independent of the choice of $\widetilde{F}$ (i.e. defines the same equivalence class of AF's for any strictly $\mathcal{E}$-q.c. ρdy-version $\widetilde{F}$ of F). The sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ implies

$$\begin{aligned}\overline{e}(A^{[F]}) &= \overline{\lim_{\alpha\to\infty}} \left(\alpha(F - \alpha G_\alpha F, F)_{\mathcal{H}} - \frac{\alpha}{2}\int_E (F^2 - \alpha G_\alpha F^2)\rho\, dy\right)\\ &\leqslant \overline{\lim_{\alpha\to\infty}}\, \alpha(F - \alpha G_\alpha F, F)_{\mathcal{H}}.\end{aligned}$$

Since $\mathcal{F} \subset \mathcal{V}^{\mathcal{F}}$ (cf. e.g. proof of [20, Lemma 3.1]) it follows $\lim_{\alpha\to\infty} \alpha\widehat{G}_\alpha F = F$ weakly in $\mathcal{V}$. Hence $\lim_{\alpha\to\infty} \alpha(F - \alpha G_\alpha F, F)_{\mathcal{H}} = \lim_{\alpha\to\infty} \mathcal{E}(F, \alpha\widehat{G}_\alpha F) = \mathcal{E}(F, F)$

whenever $F \in \mathcal{F}$. In particular

$$\overline{e}(A^{[F]}) \leqslant 2|F|^2_{\mathcal{F}} \quad \text{for any } F \in \mathcal{F}. \tag{10}$$

Define

$$\mathcal{M} = \{M | M \text{ is a finite AF, } E_y[M_t^2] < \infty, E_y[M_t] = 0$$
$$\text{for strictly } \mathcal{E}\text{-q.e. } y \in E \text{ and all } t \geqslant 0\}.$$

$M \in \mathcal{M}$ is called a *martingale additive functional* (MAF). Furthermore define

$$\overset{\circ}{\mathcal{M}} = \{M \in \mathcal{M} | \ e(M) < \infty\}.$$

The elements of $\overset{\circ}{\mathcal{M}}$ are called MAF's of finite energy.

Let A be a PCAF of $\mathbb{M}$. Its Revuz measure μ_A (see [19, Theorem 3.1]) is defined by

$$\int_E G(y)\mu_A(dy) = \lim_{\alpha \to \infty} \alpha E_{\rho\, dy}\left[\int_0^\infty e^{-\alpha t} G(Y_t)\, dA_t\right] \text{ for all } G \in \mathcal{B}^+. \tag{11}$$

The dual predictable projection $\langle M \rangle$ of the square bracket of $M \in \overset{\circ}{\mathcal{M}}$ is a PCAF of $\mathbb{M}$. It then follows from (9), (11), that one half of the total mass of the Revuz measure $\mu_{\langle M \rangle}$ is equal to the energy of M, i.e.

$$e(M) = \frac{1}{2}\int_E \mu_{\langle M \rangle}(dy). \tag{12}$$

Therefore $\mu_{\langle M \rangle}$ is also called the energy measure of M. For $M, L \in \overset{\circ}{\mathcal{M}}$ let

$$\langle M, L \rangle := \frac{1}{2}\left(\langle M + L \rangle - \langle M \rangle - \langle L \rangle\right).$$

Then $(\langle M, L \rangle_t)_{t \geqslant 0}$ is a CAF of bounded variation on each finite interval. Furthermore the finite signed measure $\mu_{\langle M,L \rangle}$ defined by $\mu_{\langle M,L \rangle} := \frac{1}{2}(\mu_{\langle M+L \rangle} - \mu_{\langle M \rangle} - \mu_{\langle L \rangle})$ is related to $\langle M, L \rangle$ in the sense of (11). If $G \in \mathcal{B}_b^+$, then $\int_E G d\mu_{\langle \cdot,\cdot \rangle}$ is symmetric, bilinear and positive on $\overset{\circ}{\mathcal{M}} \times \overset{\circ}{\mathcal{M}}$.

Define

$$\mathcal{N}_c = \{N | N \text{ is a finite CAF, } e(N) = 0, E_y[|N_t|] < \infty$$
$$\text{for strictly } \mathcal{E}\text{-q.e. } y \in E \text{ and all } t \geqslant 0\}.$$

For $F \in \mathcal{F}$, $A^{[F]}$ can uniquely be decomposed (see [19, Theorem 4.5.(i)], [22, Remark 0.17]) as

$$A^{[F]} = M^{[F]} + N^{[F]}, \quad M^{[F]} \in \overset{\circ}{\mathcal{M}}, \quad N^{[F]} \in \mathcal{N}_c. \tag{13}$$

The identity (13) means that both sides are equivalent as additive functionals w.r.t. $\mathrm{Cap}_{1,\widehat{G}_1\Phi}$. The uniqueness of (13) implies $aM^{[F]} + bM^{[G]} = M^{[aF+bG]}$, $aN^{[F]} + bN^{[G]} = N^{[aF+bG]}$, for any $a, b \in \mathbb{R}$, $F, G \in \mathcal{F}$.

Lemma 2.1. (i) *Let* $F \in \mathcal{F}$. *Then*

$$\mu_{\langle M^{[F]}\rangle}(dy) = \langle A\nabla F, \nabla F\rangle \rho\, dy - cF^2 \rho\, dy.$$

(ii) *Let* $c \equiv 0$. *Let* $F \in \mathcal{F}$ *be constant* ρdy*-a.e. on a Borel set B. Then*

$$\mu_{\langle M^{[F]}\rangle}(B) = 0.$$

Proof. (i) Let $V \in C_0^1(E)$, $0 \leqslant V \leqslant 1$, and $F \in C_0^1(E)$. Then

$$\begin{aligned}
&\lim_{\alpha\to\infty} \alpha \int_E V(y) E_y \left[\int_0^\infty e^{-\alpha t} d\langle M^{[F]}\rangle_t \right] \rho(y)\, dy \\
&= \lim_{\alpha\to\infty} \alpha^2 \int_E V(y) E_y \left[\int_0^\infty e^{-\alpha t} (M_t^{[F]})^2\, dt \right] \rho(y)\, dy \\
&= \lim_{\alpha\to\infty} \alpha^2 \int_E V(y) E_y \left[\int_0^\infty e^{-\alpha t} (F(Y_t) - F(Y_0))^2\, dt \right] \rho(y)\, dy \\
&= \lim_{\alpha\to\infty} \Big(2\alpha (F - \alpha G_\alpha F, FV)_{\mathcal{H}} - \alpha (F^2, V - \alpha \widehat{G}_\alpha V)_{\mathcal{H}} \Big) \\
&= \lim_{\alpha\to\infty} \Big(2\mathcal{E}(\alpha G_\alpha F, FV) - \mathcal{E}(F^2, \alpha \widehat{G}_\alpha V) \Big) \\
&= 2\mathcal{A}(F, FV) - \mathcal{A}(F^2, V) - 2\langle \Lambda F, FV\rangle + \langle \Lambda F^2, V\rangle \\
&= \int_E V \langle A\nabla F, \nabla F\rangle \rho\, dy - \int_E V c F^2 \rho\, dy
\end{aligned} \tag{14}$$

where the last equality follows from Lemma 1.2. Thus

$$\lim_{\alpha\to\infty} \alpha \int_E V(y) E_y \left[\int_0^\infty e^{-\alpha t} d\langle M^{[F]}\rangle_t \right] \rho(y)\, dy$$

exists and is hence by the Tauberian theorem [24, V.4.3, p. 192] equal to

$$\lim_{t\downarrow 0}\frac{1}{t}\int_E V(y)E_y\left[\langle M^{[F]}\rangle_t\right]\rho(y)\,dy.$$

On the other hand

$$\lim_{\alpha\to\infty}\alpha\int_E E_y\left[\int_0^\infty e^{-\alpha t}V(Y_t)\,d\langle M^{[F]}\rangle_t\right]\rho(y)\,dy$$

exists as a bounded and increasing limit and equals hence by [24, V.4.3, p. 192]

$$\lim_{t\downarrow 0}\frac{1}{t}\int_E E_y\left[\int_0^t V(Y_s)\,d\langle M^{[F]}\rangle_s\right]\rho(y)\,dy.$$

Let $(p_t)_{t\geqslant 0}$ be the transition semigroup, and $(\vartheta_t)_{t\geqslant 0}$ be the shift operator of $\mathbb{M}$. Note that $t\mapsto V(Y_t)$ is continuous by Theorem 1.9. Using the equalities of the limits just above, Lebesgue's theorem, the Markov property, the p_t-subinvariance of ρdy, and (14), we obtain

$$\begin{aligned}
&\int_E V(y)\mu_{\langle M^{[F]}\rangle}(dy)\\
&= \lim_{\alpha\to\infty}\alpha\int_E E_y\left[\int_0^\infty e^{-\alpha t}V(Y_t)\,d\langle M^{[F]}\rangle_t\right]\rho(y)\,dy\\
&= \lim_{t\downarrow 0}\frac{1}{t}\int_E E_y\left[\int_0^t V(Y_s)\,d\langle M^{[F]}\rangle_s\right]\rho(y)\,dy\\
&= \lim_{t\downarrow 0}\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{t}\int_E E_y\left[V(Y_{k\frac{t}{n}})\left(\langle M^{[F]}\rangle_{(k+1)\frac{t}{n}}-\langle M^{[F]}\rangle_{k\frac{t}{n}}\right)\right]\rho(y)\,dy\\
&= \lim_{t\downarrow 0}\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{t}\int_E E_y\left[V(Y_{k\frac{t}{n}})\left(\langle M^{[F]}\rangle_{\frac{t}{n}}\circ\vartheta_{k\frac{t}{n}}\right)\right]\rho(y)\,dy\\
&= \lim_{t\downarrow 0}\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{t}\int_E p_{k\frac{t}{n}}\left(V(\cdot)E_\cdot\left[\langle M^{[F]}\rangle_{\frac{t}{n}}\right]\right)(y)\rho(y)\,dy\\
&\leqslant \lim_{t\downarrow 0}\lim_{n\to\infty}\frac{n}{t}\int_E V(y)E_y\left[\langle M^{[F]}\rangle_{\frac{t}{n}}\right]\rho(y)\,dy\\
&= \lim_{t\downarrow 0}\frac{1}{t}\int_E V(y)E_y\left[\langle M^{[F]}\rangle_t\right]\rho(y)\,dy\\
&= \lim_{\alpha\to\infty}\alpha\int_E V(y)E_y\left[\int_0^\infty e^{-\alpha t}d\langle M^{[F]}\rangle_t\right]\rho(y)\,dy
\end{aligned}$$

$$= \int_E V\langle A\nabla F, \nabla F\rangle \rho\, dy - \int_E VcF^2\rho\, dy. \tag{15}$$

It follows that $\mathrm{supp}(\mu_{\langle M^{[F]}\rangle}(dy)) \subset \mathrm{supp}(F)$. Let $W \in C_0^1(E)$, $V \leqslant W \leqslant 1$, $W = 1$ on $\mathrm{supp}(F)$. Then $W - V \in C_0^1(E)$, $0 \leqslant W - V \leqslant 1$. Hence by (15)

$$\int_E (W - V)\, d\mu_{\langle M^{[F]}\rangle} \leqslant \int_E (W - V)\langle A\nabla F, \nabla F\rangle \rho\, dy - \int_E (W - V)cF^2\rho\, dy. \tag{16}$$

We note that $FW = F$ and $\alpha G_\alpha F^2 \geqslant W\alpha G_\alpha F^2$ for any α. We have

$$\begin{aligned}
\int_E W d\mu_{\langle M^{[F]}\rangle} &= \int_E d\mu_{\langle M^{[F]}\rangle} \\
&= \lim_{\alpha\to\infty} \left(2\alpha(F - \alpha G_\alpha F, F)_{\mathcal{H}} - \alpha \int (F^2 - \alpha G_\alpha F^2)\rho\, dy\right) \\
&\geqslant \lim_{\alpha\to\infty} \left(2\alpha(F - \alpha G_\alpha F, FW)_{\mathcal{H}} - \alpha \int W(F^2 - \alpha G_\alpha F^2)\rho\, dy\right) \\
&= \int_E W\langle A\nabla F, \nabla F\rangle \rho\, dy - \int_E WcF^2\rho\, dy.
\end{aligned} \tag{17}$$

Subtracting (16) from (17) gives

$$\int_E V\, d\mu_{\langle M^{[F]}\rangle} \geqslant \int_E V\langle A\nabla F, \nabla F\rangle \rho\, dy - \int_E VcF^2\rho\, dy$$

and then equality because of (15). The assertion now easily follows for $F \in C_0^1(E)$. For general $F \in \mathcal{F}$ let $(F_n)_{n\in\mathbb{N}} \subset C_0^1(E)$, $\lim_{n\to\infty} F_n = F$ in $\mathcal{F}$. Let $G \in \mathcal{B}_b^+$. Then by the Cauchy–Schwarz inequality, (12), (10),

$$\begin{aligned}
&\left|\int_E G\, d\mu_{\langle M^{[F]}\rangle} - \int_E G\, d\mu_{\langle M^{[F_n]}\rangle}\right| = \left|\int_E G\, d\mu_{\langle M^{[F+F_n]}, M^{[F-F_n]}\rangle}\right| \\
&\leqslant |G|_\infty \left(\int_E d\mu_{\langle M^{[F+F_n]}\rangle}\right)^{\frac{1}{2}} \left(\int_E d\mu_{\langle M^{[F-F_n]}\rangle}\right)^{\frac{1}{2}} \\
&= 2|G|_\infty \left(e(M^{[F+F_n]})\right)^{\frac{1}{2}} \left(e(M^{[F-F_n]})\right)^{\frac{1}{2}} \\
&\leqslant 2|G|_\infty |F + F_n|_{\mathcal{F}} |F - F_n|_{\mathcal{F}}.
\end{aligned}$$

Since the last term converges to zero as n tends to infinity the assertion follows.

(ii) follows easily from (i). $\square$

A (positive) Borel measure ν on E is called *smooth* w.r.t. $(\mathcal{A}, \mathcal{V})$, if:
(i) ν does not charge sets of zero $\mathrm{Cap}^{\mathcal{A}}$-capacity,
(ii) there exists an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E such that

$$\nu(F_k) < \infty, \quad k = 1, 2, \ldots,$$

$$\lim_{n\to\infty} \mathrm{Cap}^{\mathcal{A}}(K \setminus F_n) = 0 \text{ for any compact set } K \subset E.$$

There are several equivalent formulations of smoothness (see [6, Theorem 2.2.4]).

If ℓ is a PCAF of $\mathbb{M}$, $F \in \mathcal{B}^+$, $\alpha > 0$, as usual we set

$$U_\ell^\alpha F(s,x) := E_{(s,x)}\left[\int_0^\infty e^{-\alpha t} F(Y_t)\, d\ell_t\right].$$

The following is fundamental. It tells us in particular that if the space derivative drift part obtained through integration by parts in $\mathcal{A}$ is a "good" measure w.r.t. $|\cdot|_{\mathcal{V}}$, then it may be represented through a positive continuous additive functional.

Theorem 2.2. *Let ν be a positive Borel measure on E. Assume that there exists a strict $\mathcal{E}$-nest $(\overline{E}_k)_{k\geqslant 1}$ such that $1_{\overline{E}_k} d\nu$ is a finite smooth measure w.r.t. $(\mathcal{A}, \mathcal{V})$. Then there exists (up to $\mathrm{Cap}_{1,\widehat{G}_1\Phi}$-equivalence) a unique positive continuous additive functional ℓ of $\mathbb{M}$ such that*

$$\int_E \widetilde{V} F d\nu = \lim_{\alpha\to\infty} \int_E V \alpha U_\ell^{\alpha+1} F \rho\, dy. \tag{18}$$

for any $\mathcal{A}$-q.c. ρdy-version $\widetilde{V}$ of $V \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}} \cap \mathcal{V}_b^{\mathcal{F}}$, and any $F \in \mathcal{B}_b^+$.

Proof. For $F \in \mathcal{V}$ throughout this proof $\widetilde{F}$ will denote an $\mathcal{A}$-q.c. ρdy-version of F.

For $L \in \mathbb{N}$, let $\nu_L := 1_{\overline{E}_L} d\nu$. By Fukushima et al. [6, Theorem 2.2.4] there exists an $\mathcal{A}$-nest $(F_k)_{k\in\mathbb{N}}$ on E and for each $k, L \in \mathbb{N}$ a constant $C_{k,L}$ such that

$$\int_E |\widetilde{F}|\, 1_{F_k} d\nu_L \leqslant C_{k,L} |F|_{\mathcal{V}}$$

for any $\mathcal{A}$-quasi-continuous ρdy-version $\widetilde{F}$ of $F \in \mathcal{V}$. For $L, N \in \mathbb{N}$, let $\mu_{L,N} := 1_{F_N} d\nu_L$, and $v_{U_{L,N}} \in \mathcal{V}$ be the 1-potential of $\mu_{L,N}$ w.r.t. $(\mathcal{A}, \mathcal{V})$.

If $\widetilde{\widehat{G}_1 F}$ denotes an $\mathcal{A}$-quasi-continuous ρdy-version of $\widehat{G}_1 F$, $F \in \mathcal{V}'$, then

$$I_{L,N}(F) := \int_E \widetilde{\widehat{G}_1 F} d\mu_{L,N}$$

is a well-defined continuous linear functional on $\mathcal{V}'$. Indeed, denoting by $|\widehat{G}_1|_{L(\mathcal{V}')}$ the operator norm of $\widehat{G}_1$, we have

$$I_{L,N}(F) \leqslant C_{L,N}|\widehat{G}_1 F|_{\mathcal{V}} \leqslant C_{L,N}|\widehat{G}_1|_{L(\mathcal{V}')}|F|_{\mathcal{V}'}.$$

There exists hence a unique $U_{L,N} \in \mathcal{V}'' = \mathcal{V}$ with

$$\mathcal{E}_1(U_{L,N}, \widehat{G}_1 F) = \langle U_{L,N}, F\rangle = \int_E \widetilde{\widehat{G}_1 F}\, d\mu_{L,N} = \mathcal{A}_1(v_{U_{L,N}}, \widehat{G}_1 F)$$

for any $F \in \mathcal{V}'$. Of course, $U_{L,N}$ is 1-excessive w.r.t. $(G_\alpha)_{\alpha>0}$.

We make a short remark. Let μ be of finite 1-order energy integral w.r.t. $(\mathcal{A}, \mathcal{V})$. Because of the inequality

$$\left|\int_E F\, d\mu\right| \leqslant C|F|_{\mathcal{V}} \leqslant C|F|_{\mathcal{F}}; \quad F \in C_0^1(E),$$

and the fact that $C_0^1(E)$ is dense both in $\mathcal{V}$ and in $\mathcal{F}$, we can conclude that any measure of finite energy integral w.r.t. $(\mathcal{A}, \mathcal{V})$ is of finite 1-coenergy integral w.r.t. $(\mathcal{E}, \mathcal{F})$. Hence any strictly $\mathcal{E}$-exceptional set is also $\mathcal{A}$-exceptional.

Since $U_{L,N} \in \mathcal{V}$ and $(G_\alpha)_{\alpha>0}$ is strongly continuous on $\mathcal{V}$ we may assume that there exists an $\mathcal{A}$-nest $(K_k)_{k\in\mathbb{N}}$ and an $\mathcal{A}$-q.c. ρdy-version $\widetilde{U}_{L,N}$ of $U_{L,N}$ such that

$$nR_{n+1}U_{L,N}(y) \to \widetilde{U}_{L,N}(y) \text{ uniformly on } K_k \text{ for any } k \in \mathbb{N} \text{ as } n \to \infty.$$

In particular $\widetilde{U}_{L,N}(y) = \overline{U}_{L,N}(y) := \sup_{\alpha>0} \alpha R_{\alpha+1} U_{L,N}(y)$ strictly $\mathcal{E}$-q.e. Let $\overline{K}_k := K_k \cap F_k$, $k \in \mathbb{N}$. Then $(\overline{K}_k)_{k\in\mathbb{N}}$ is again an $\mathcal{A}$-nest and $\overline{\mu}_{L,N} := 1_{\overline{K}_N} d\nu_L = 1_{K_N} d\mu_{L,N}$ is again of finite 1-order energy integral w.r.t. $(\mathcal{A}, \mathcal{V})$. Let $v_{V_{L,N}} \in \mathcal{V}$ be its 1-potential w.r.t. $(\mathcal{A}, \mathcal{V})$. As before there is a unique element in $V_{L,N} \in \mathcal{V}$ such that

$$\mathcal{E}_1(V_{L,N}, F) = \int_E \widetilde{F}\, d\overline{\mu}_{L,N} = \mathcal{A}_1(v_{V_{L,N}}, F) \tag{19}$$

for any $\mathcal{A}$-quasi-continuous ρdy-version $\widetilde{F}$ of $F \in \widehat{\mathcal{F}} \subset \mathcal{V}$. The r.h.s. equality holding for any $F \in \mathcal{V}$. Let $\overline{V}_{L,N}(y) := \sup_{\alpha>0} \alpha R_{\alpha+1} V_{L,N}(y)$. For $M \leqslant N$, $n \geqslant 1$, we have

$$\overline{V}_{L,M} - nR_{n+1}V_{L,M} \leqslant \overline{V}_{L,N} - nR_{n+1}V_{L,N} \leqslant \overline{U}_{L,N} - nR_{n+1}U_{L,N}$$

strictly $\mathcal{E}$-q.e. in E. We only show the second inequality. The first follows similarly. Indeed, for any positive $H \in \mathcal{H}$

$$(V_{L,N} - nR_{n+1}V_{L,N}, H)_{\mathcal{H}} = \int_E \widetilde{\widehat{G}_{n+1}H}\, d\overline{\mu}_{L,N} \leqslant \int_E \widetilde{\widehat{G}_{n+1}H}\, d\mu_{L,N}$$
$$= (U_{L,N} - nR_{n+1}U_{L,N}, H)_{\mathcal{H}}.$$

Hence $V_{L,N} - nR_{n+1}V_{L,N} \leqslant U_{L,N} - nR_{n+1}U_{L,N}$ $\rho(y)dy$-a.e. and then strictly $\mathcal{E}$-q.e. We further show

$$\overline{V}_{L,N}(y) = E_y[e^{-\sigma_{\overline{K}_N}}\overline{V}_{L,N}(Y_{\sigma_{\overline{K}_N}})] \quad \text{for strictly} \quad \mathcal{E}\text{-q.e. } y \in E.$$

In order to do this we let $(U_l^N)_{l\in\mathbb{N}}$ be open sets containing $\overline{K}_N$ and such that $\cap_{l\in\mathbb{N}} U_l^N = \overline{K}_N$. Since $\mathbb{M}$ is a Hunt process we have $P_y(\lim_{l\to\infty} Y_{\sigma_{U_l^N}} = Y_{D_{\overline{K}_N}}) = 1$ for any $y \in E$. Note that additionally $P_y(D_{\overline{K}_N} = \sigma_{\overline{K}_N}) = 1$ for ρdy-a.e. $y \in E$. Let $H \in \mathcal{H}$ be positive. For $U \subset E$, U open let $(\widehat{G}_1 H)_U$ be the 1-coreduced function on U w.r.t. $\mathcal{E}$. Since $\widehat{G}_1 H$, $(\widehat{G}_1 H)_U \in \mathcal{V}$ we know that there exist $\mathcal{A}$-q.c. ρdy-versions of $\widehat{G}_1 H$, $(\widehat{G}_1 H)_U$, which we denote by $\widetilde{\widehat{G}_1 H}$, $\widetilde{(\widehat{G}_1 H)}_U$. Since $\widehat{G}_1 H = (\widehat{G}_1 H)_U$ ρdy-a.e. on U it follows $\widetilde{\widehat{G}_1 H} = \widetilde{(\widehat{G}_1 H)}_U$ $\mathcal{A}$-q.e. U. For $\alpha > 0$ let $(\widehat{G}_1 H)_U^\alpha$ denote the unique solution $F^\alpha \in \widehat{\mathcal{F}}$ to the equation $\mathcal{E}_1(V, F^\alpha) = \alpha((F^\alpha - \widehat{G}_1 H \cdot 1_U)^-, V)_{\mathcal{H}}$, $\forall V \in \mathcal{V}$. We know that $\lim_{\alpha\to\infty} (\widehat{G}_1 H)_U^\alpha = (\widehat{G}_1 H)_U$ weakly in $\mathcal{V}$. Finally

$$(V_{L,N}, H)_{\mathcal{H}} = \lim_{l\to\infty} \int_E \widetilde{(\widehat{G}_1 H)}_{U_l^N} d\overline{\mu}_{L,N} = \lim_{l\to\infty} \lim_{\alpha\to\infty} \mathcal{A}_1(v_{V_{L,N}}, (\widehat{G}_1 H)^\alpha_{U_l^N})$$
$$= \lim_{l\to\infty} \lim_{\alpha\to\infty} \mathcal{E}_1(V_{L,N}, (\widehat{G}_1 H)^\alpha_{U_l^N}) = (E[e^{-\sigma_{\overline{K}_N}}\overline{V}_{L,N}(Y_{\sigma_{\overline{K}_N}})], H)_{\mathcal{H}}.$$

Hence $\overline{V}_{L,N} = E[e^{-\sigma_{\overline{K}_N}}\overline{V}_{L,N}(Y_{\sigma_{\overline{K}_N}})]$ ρdy-a.e. and then strictly $\mathcal{E}$-q.e. For well-chosen strictly $\mathcal{E}$-exceptional sets N_1, N_2, N_3, N_4, the last implies

$$\sup_{y\in E\setminus N_1} \left(\overline{V}_{L,N} - nR_{n+1}V_{L,N}\right)(y)$$
$$\leqslant \sup_{y\in E\setminus N_2} \left(E.[e^{-\sigma_{\overline{K}_N}}\overline{V}_{L,N}(Y_{\sigma_{\overline{K}_N}})] - E.[e^{-\sigma_{\overline{K}_N}} nR_{n+1}V_{L,N}(Y_{\sigma_{\overline{K}_N}})]\right)(y)$$
$$\leqslant \sup_{y\in \overline{K}_N\setminus N_3} \left(\overline{V}_{L,N} - nR_{n+1}V_{L,N}\right)(y)$$
$$\leqslant \sup_{y\in \overline{K}_N\setminus N_4} \left(\overline{U}_{L,N} - nR_{n+1}U_{L,N}\right)(y).$$

But $nR_{n+1}U_{L,N}(y) \to \widetilde{U}_{L,N}(y)$ uniformly on $\overline{K}_N$ as $n \to \infty$. Hence $nR_{n+1}V_{L,N} \to \overline{V}_{L,N}$ uniformly on $E \setminus N_1$ as $n \to \infty$. Now, let us define

$$\ell_t^{L,N,n}(\omega) := \int_0^t e^{-s} n(\overline{V}_{L,N} - nR_{n+1}V_{L,N})(Y_s(\omega))\, ds, \quad 0 \leqslant t \leqslant \infty.$$

Let $\widehat{v} \in \widehat{S}_{00}^{\text{str}}$, $\widehat{v}(E) = 1$. Then

$$M_t^n := \ell_t^{L,N,n} + e^{-t} nR_{n+1}V_{L,N}(Y_t), \quad 0 \leqslant t \leqslant \infty,$$

is an $((\mathcal{F}_t)_{t \geqslant 0}, P_{\widehat{v}})$-martingale. Using Doob's inequality it follows

$$P_{\widehat{v}}\left(\sup_{0 \leqslant t \leqslant \infty} |M_t^n - M_t^l| > \varepsilon\right) \leqslant \varepsilon^{-2} E_{\widehat{v}}[(\ell_\infty^{L,N,n} - \ell_\infty^{L,N,l})^2].$$

A standard calculation together with our above achievements show, that for $n > l$, and some constant $C_{\widehat{v}}$

$$\begin{aligned} E_{\widehat{v}}[(\ell_\infty^{L,N,n} - \ell_\infty^{L,N,l})^2] &\leqslant C_{\widehat{v}}\, \mathcal{E}_1(nR_{n+1}V_{L,N}, V_{L,N} - lR_{l+1}V_{L,N}) \\ &\leqslant \sup_{y \in \overline{K}_N \setminus N_1} \left(\overline{V}_{L,N} - nR_{n+1}V_{L,N}\right)(y)\, C_{\widehat{v}}\, \overline{\mu}_{L,N}(E). \end{aligned}$$

Hence, $(\ell_\infty^{L,N,n})_{n \in \mathbb{N}}$ is a $L^2(P_{\widehat{v}})$-Cauchy sequence and we can chose a subsequence which we again denote by n such that

$$P_{\widehat{v}}(M_t^n \text{ converges uniformly on } [0, \infty) \text{ as } n \to \infty) = 1.$$

On the other hand since $nR_{n+1}V_{L,N}$ converges uniformly to $\overline{V}_{L,N}$ we have

$$P_{\widehat{v}}(e^{-t} nR_{n+1}V_{L,N}(Y_t) \text{ converges uniformly on } [0, \infty) \text{ as } n \to \infty) = 1.$$

Hence

$$\ell_t^{L,N} := \lim_{n \to \infty} \ell_t^{L,N,n}$$

defines a PCAF of $\mathbb{M}$. Furthermore, since $\ell_t^{L,N} \leqslant \ell_t^{L,M}$ if $N \leqslant M$, we know that $\ell_t^{L,M} - \ell_t^{L,N}$ is a PCAF and therefore it increases. Hence

$$E_{\widehat{v}}\left[\sup_{t \geqslant 0} |\ell_t^{L,M} - \ell_t^{L,N}|\right] \leqslant E_{\widehat{v}}[\ell_\infty^{L,M} - \ell_\infty^{L,N}] \leqslant \int_E \widetilde{\widehat{G}_1 H 1_{\overline{K}_N^c}}\, dv_L$$

if $H \in \mathcal{H}_b$ is such that $\widehat{U}_1\widehat{v} \leqslant \widehat{G}_1 H$. The last term converges to zero as $M \geqslant N \to \infty$. Hence

$$\ell_t^L := \lim_{N\to\infty} \ell_t^{L,N}$$

defines again a PCAF of $\mathbb{M}$. Of course, we also have $\ell_t^{L,N} \leqslant \ell_t^{K,N}$, if $L \leqslant K$, and $\ell^L \leqslant \ell^K$, if $L \leqslant K$.

Let $V \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}} \cap \mathcal{V}_b^{\mathcal{F}}$, $G \in C_0^1(E)$, $G \geqslant 0$. We can apply [20, (19)] to see that $\alpha\widehat{G}_{\alpha+1}(V)\,G \in \mathcal{V}^{\mathcal{F}}$. It follows that $\lim_{n\to\infty} n\widehat{G}_{n+1}(\alpha\widehat{G}_{\alpha+1}(V)\,G) = \alpha\widehat{G}_{\alpha+1}(V)\,G$ weakly in $\mathcal{V}$. Then by (19)

$$\begin{aligned}
\lim_{\alpha\to\infty} \int_E V\alpha U^{\alpha}_{\ell^{L,N}} G\,\rho\,dy &= \lim_{\alpha\to\infty}\lim_{n\to\infty} \int_E V\alpha U^{\alpha+1}_{\ell^{L,N,n}} G\,\rho\,dy \\
&= \lim_{\alpha\to\infty}\lim_{n\to\infty} \int_E \alpha\widehat{G}_{\alpha+1}(V)\,G\,n(V_{L,N} - nG_{n+1}V_{L,N})\rho\,dy \\
&= \lim_{\alpha\to\infty}\lim_{n\to\infty} \mathcal{A}_1\left(v_{V_{L,N}}, n\widehat{G}_{n+1}(\alpha\widehat{G}_{\alpha+1}(V)\,G)\right) \\
&= \lim_{\alpha\to\infty} \int_E \alpha\widetilde{\widehat{G}_{\alpha+1}(V)}G\,d\overline{\mu}_{L,N} = \int_E \widetilde{V}G d\overline{\mu}_{L,N}.
\end{aligned}$$

Noting that $\int_E V\alpha U^{\alpha+1}_{\ell^{L,N}} G\,\rho dy$ increases in α we easily see that

$$\lim_{\alpha\to\infty} \int_E V\alpha U^{\alpha+1}_{\ell^{L,N}} G\,\rho\,dy = \int_E \widetilde{V}G d\overline{\mu}_{L,N} \tag{20}$$

for any $V \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}} \cap \mathcal{V}_b^{\mathcal{F}}$, $G \in \mathcal{B}_b^+(E)$. Since $\ell^{L,N}$ is increasing in N, and $(\overline{K}_N)_{N\in\mathbb{N}}$ is an $\mathcal{A}$-nest, we obtain letting $N \to \infty$

$$\lim_{\alpha\to\infty} \int_E V\alpha U^{\alpha+1}_{\ell^{L}} G\,\rho\,dy = \int_E \widetilde{V}G\,d\nu_L. \tag{21}$$

Because by Trutnau [19, Theorem 3.1], [22, Remark 0.17], any PCAF a_t of $\mathbb{M}$ admits a strict $\mathcal{E}$-nest $(E_k)_{k\in\mathbb{N}}$ such that $U_a^1 1_{E_k} \in \mathcal{P}_{G_1\mathcal{H}_b^+}$, we can use standard arguments to see that any PCAF of $\mathbb{M}$ satisfying (21) is unique. Hence (21) implies that $(1_{\overline{E}_N}\cdot\ell^L)_t = \ell_t^N$, if $L \geqslant N$. We now define a PCAF ℓ of $\mathbb{M}$ through

$$\ell_t(\omega) := \begin{cases} \ell_t^L(\omega) & \text{if } \sigma_{\overline{E}^c_{L-1}}(\omega) \leqslant t < \sigma_{\overline{E}^c_L}(\omega),\ \ L = 2, 3, \dots \\ \ell_{\sigma(\omega)-}(\omega) & \text{if } t \geqslant \sigma(\omega) := \lim_{L\to\infty} \sigma_{\overline{E}^c_L}(\omega). \end{cases}$$

It follows that $\ell_t = \ell_t^L$ for $t < \sigma_{\overline{E}_L^c}$, hence $\ell_t = \lim_{L\to\infty} \ell_t^L$ for $t < \zeta$. Since ℓ^L is increasing in L we obtain using again (21) the statement of the theorem. □

Theorem 2.3. *For $F \in \mathcal{F}$ the following are equivalent*:

(i) $A \in \mathcal{N}_c$ *and*

$$\lim_{\alpha\to\infty} \alpha^2 E_{\widehat{G}_1 W \rho\, dy}\left[\int_0^\infty e^{-\alpha t} A_t\, dt\right] = -\mathcal{E}(F, \widehat{G}_1 W) \ \ for\ all\ \ W \in \mathcal{H}_b.$$

(ii) $A = N^{[F]}$.

Proof (cf. Fukushima et al. [6, Theorem 5.2.4], Oshima [10, Theorem 5.2.5]). (ii) $\Rightarrow$ (i): If $A = N^{[F]}$ then $A \in \mathcal{N}_c$ and

$$\begin{aligned}
&\lim_{\alpha\to\infty} \alpha^2 E_{\widehat{G}_1 W \rho\, dy}\left[\int_0^\infty e^{-\alpha t}(\widetilde{F}(Y_t) - \widetilde{F}(Y_0))\, dt\right] \\
&\quad = \lim_{\alpha\to\infty} \alpha^2 \int_E \widehat{G}_1 W \left(G_\alpha F - \frac{1}{\alpha} F\right) \rho\, dy \\
&\quad = \lim_{\alpha\to\infty} \alpha(\alpha G_\alpha F - F, \widehat{G}_1 W)_{\mathcal{H}} \\
&\quad = \lim_{\alpha\to\infty} -\mathcal{E}(F, \alpha \widehat{G}_\alpha \widehat{G}_1 W) = -\mathcal{E}(F, \widehat{G}_1 W)
\end{aligned}$$

for all $W \in \mathcal{H}_b$.

(i) $\Rightarrow$ (ii): If $A \in \mathcal{N}_c$ then $E_y[A_t] \in \mathcal{H}$,

$$c_\alpha(y) := E_y\left[\int_0^\infty e^{-\alpha t} A_t\, dt\right] \in \mathcal{H}, \alpha > 0,$$

and for $\alpha \neq \beta > 0$

$$\begin{aligned}
R_\beta(\alpha c_\alpha) &= \alpha E_\cdot\left[\int_0^\infty e^{-\beta s} E_{Y_s}\left[\int_0^\infty e^{-\alpha t} A_t\, dt\right] ds\right] \\
&= \alpha E_\cdot\left[\int_0^\infty \int_0^\infty e^{-\beta s} e^{-\alpha t}(A_{t+s} - A_s)\, dt\, ds\right] \\
&= \alpha E_\cdot\left[\int_0^\infty \int_s^\infty e^{-(\beta-\alpha)s} e^{-\alpha t} A_t\, dt\, ds\right] - E_\cdot\left[\int_0^\infty e^{-\beta s} A_s\, ds\right] \\
&= \alpha E_\cdot\left[\int_0^\infty e^{-\alpha t} \int_0^t e^{-(\beta-\alpha)s}\, ds A_t\, dt\right] - c_\beta
\end{aligned}$$

$$=\frac{\alpha}{\alpha-\beta}(c_\beta - c_\alpha) - c_\beta = \frac{1}{\alpha-\beta}(\beta c_\beta - \alpha c_\alpha).$$

Hence

$$R_\beta(\alpha c_\alpha) = R_\alpha(\beta c_\beta).$$

Note that $\widehat{G}_\beta W = \widehat{G}_1(W - (\beta-1)\widehat{G}_\beta W)$ and therefore

$$\begin{aligned}(\beta c_\beta, W)_{\mathcal{H}} &= \lim_{\alpha\to\infty}(\alpha G_\alpha(\beta c_\beta), W)\\ &= \lim_{\alpha\to\infty}(\alpha^2 c_\alpha, \widehat{G}_\beta W) = -\mathcal{E}(F, \widehat{G}_\beta W) = (\beta G_\beta F - F, W)_{\mathcal{H}}\end{aligned}$$

for any $W \in \mathcal{H}_b$. It follows $\beta c_\beta = \beta G_\beta F - F$ for any $\beta > 0$, which further implies that

$$E_\cdot[M_t] = 0 \ \rho\, dy\text{-a.e.}, \ \ M_t := A_t - N_t^{[F]}, t \geqslant 0.$$

Thus $R_\alpha E_\cdot[M_t] = \int_0^\infty e^{-\alpha s} E_\cdot\left[E_{Y_s}[M_t]\right] ds = 0$ strictly $\mathcal{E}$-q.e. for any $\alpha > 0$. Since $s \mapsto E_y\left[E_{Y_s}[M_t]\right]$ is continuous for strictly $\mathcal{E}$-q.e. $y \in E$ by Theorem 1.9, we get $E_\cdot\left[E_{Y_s}[M_t]\right] = 0$ strictly $\mathcal{E}$-q.e., and therefore

$$E_\cdot\left[M_{t+s}\right] = E_\cdot[M_t] + E_\cdot\left[E_{Y_s}[M_t]\right] = E_\cdot[M_t] \ \text{ strictly } \mathcal{E}\text{-q.e., } t, s \geqslant 0.$$

Hence

$$E_\cdot[M_t] = 0 \text{ strictly } \mathcal{E}\text{-q.e.}$$

The last implies that M_t is a MAF, because

$$E_y\left[M_{t+s}|\mathcal{F}_s\right] = E_y\left[M_t \circ \vartheta_s + M_s|\mathcal{F}_s\right] = E_{Y_s}[M_t] + M_s = M_s$$

P_y-a.s. for strictly $\mathcal{E}$-q.e. $y \in E$. The assertion now follows because a MAF of zero energy is zero (cf. [21, Theorem 4.4]). □

Let $\overline{\rho} \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$ and let $(A^{\overline{\rho}}, D(A^{\overline{\rho}}))$ be a Dirichlet form on $L^2(D, \overline{\rho}\, dx)$ with the following properties: It holds

$$E_t \subset D \subset \mathbb{R}^d, \quad \forall t \in \mathbb{R},$$

and $(A^{\overline{\rho}}, D(A^{\overline{\rho}}))$ is the closure of

$$\frac{1}{2}\sum_{i=1}^{d}\int_D \partial_i\phi\,\partial_i\psi\,\overline{\rho}\,dx; \quad \phi,\psi\in C_0^\infty(\mathbb{R}^d)|_D,$$

on $L^2(D,\overline{\rho}\,dx)$.

Lemma 2.4. (i) *Suppose* $(A1)$, $(A2)$, $(A3)$, *and* $(A4)$. *Let* $c\equiv 0$, $\kappa(s,t)=s\pm t$, *or* $\kappa(s,t)=se^{dt}$, *d a positive constant. Assume that there is* $h\in L^1_{\text{loc}}(T_I,ds)$, $\overline{\rho}\in L^1_{\text{loc}}(\mathbb{R}^d,dx)$, *such that* $(A^{\overline{\rho}}, D(A^{\overline{\rho}}))$ *is recurrent, and such that* $\rho(s,x)\leqslant h(s)\overline{\rho}(x)\,dx$ ds*-a.e. Then the resolvent* $(G_\alpha)_{\alpha>0}$ *associated with* $\mathcal{E}$ *is Markovian.*

(ii) $(A1')$, $(A2)$, $(A3)$, *and* $(A4)$. *Let* $c\equiv 0$, $\kappa(s,t)=s\pm t$, *or* $\kappa(s,t)=se^{dt}$, *d a positive constant. Assume that there is* $h\in L^1_{\text{loc}}(T_I,ds)$, $\overline{\rho}\in L^1_{\text{loc}}(\mathbb{R}^d,dx)$, *such that* $(A^{\overline{\rho}}, D(A^{\overline{\rho}}))$ *is recurrent, and such that* $a_{ij}\rho(s,x)\leqslant h(s)\overline{\rho}(x)\,dx\,ds$*-a.e.,* $1\leqslant i,j\leqslant d$. *Then the resolvent* $(G_\alpha)_{\alpha>0}$ *associated with* $\mathcal{E}$ *is Markovian.*

Proof. We start this proof with a general observation. In order to prove the conservativity of a generalized Dirichlet form it is enough to show that for one $F\in\mathcal{H}\cap L^1(E,\rho dy)$, $F>0$ ρdy-a.e., there exists $(W_n)_{n\in\mathbb{N}}\subset\mathcal{F}$, $0\leqslant W_n\leqslant 1$, $n\in\mathbb{N}$, $W_n\uparrow 1$ as $n\to\infty$, such that

$$\lim_{n\to\infty}\mathcal{E}(W_n,\widehat{G}_1F)=0.$$

Indeed, if this is the case then

$$0=\lim_{n\to\infty}\mathcal{E}(W_n,\widehat{G}_1F)=\lim_{n\to\infty}\int_E(W_n-G_1W_n)F\rho\,dy=\int_E(1-G_11)F\rho\,dy$$

and $G_11=1$ as desired.

We only show the statements (i), (ii), when $\kappa(s,t)=se^{dt}$. The other statements can be shown similarly.

(i) Since $(A^{\overline{\rho}}, D(A^{\overline{\rho}}))$ is recurrent, there exists (see [6, Theorem 1.6.6]) $(\psi_n)_{n\in\mathbb{N}}\subset D(A^{\overline{\rho}})$, $0\leqslant\psi_n\leqslant 1$, $\psi_n\uparrow 1$, with $\lim_{n\to\infty}A^{\overline{\rho}}(\psi_n,\psi_n)=0$. We may assume $A^{\overline{\rho}}(\psi_n,\psi_n)=(\int_{[-2n,2n]\cap T_I}h(s)ds)^{-1}\frac{1}{n}$. Let $g_n\in C_0^1(\mathbb{R})$, $n\in\mathbb{N}$, $supp(g_n)\subset[-2n,2n]$, $g_n\equiv 1$ on $[-n,n]$, $|\partial_t g_n|_\infty\leqslant\frac{2}{n}$ on $[-2n,-n]\cup[n,2n]$. Obviously $W_n:=g_n\psi_{n|E}\in\mathcal{F}$. Then,

$$\begin{aligned}&\left|\mathcal{E}(W_n,\widehat{G}_1F)\right|\\&\quad\leqslant\mathcal{A}(\widehat{G}_1F,\widehat{G}_1F)^{1/2}\left(C\,\frac{1}{2}\sum_{i=1}^{d}\int_{T_I}g_n^2(s)\int_{E_s}\left(\partial_i\psi_n(x)\right)^2\rho(s,x)\,dx\,ds\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_{T_I}\int_{E_s} sd\partial_t(g_n\psi_n)(s,x)\widehat{G}_1F(s,x)\rho(s,x)\,dx\,ds\right| \\
&\leqslant \mathcal{A}(\widehat{G}_1F,\widehat{G}_1F)^{\frac{1}{2}}\left(C\int_{[-2n,2n]\cap T_I} h(s)\,ds\,A^{\overline{\rho}}(\psi_n,\psi_n)\right)^{\frac{1}{2}} \\
&+\int_{T_I\cap[-2n,2n]}\int_{E_s}\left|sd\partial_t g_n(s,x)\right|\widehat{G}_1F(s,x)\rho(s,x)\,dx\,ds \\
&\leqslant \mathcal{A}(\widehat{G}_1F,\widehat{G}_1F)^{\frac{1}{2}}\left(C\,\frac{1}{n}\right)^{\frac{1}{2}} \\
&+4d\int_E\left(1_{[-2n,-n]}(s)+1_{[n,2n]}(s)\right)\widehat{G}_1F(s,x)\rho(s,x)\,dx\,ds
\end{aligned}
$$

and the assertion follows, because $\widehat{G}_1F\in L^1(E,\rho dy)$.

(ii) We maintain the same notations as in (i). The time derivative part of $\mathcal{E}$ can be treated as in (i). For the space derivative part of $\mathcal{E}$ we calculate

$$
\begin{aligned}
\mathcal{A}(W_n,\widehat{G}_1F) &\leqslant \mathcal{A}(\widehat{G}_1F,\widehat{G}_1F)^{\frac{1}{2}}\left(\frac{1}{2}\sum_{i,j=1}^{d}\int_{T_I} g_n^2(s)\right. \\
&\qquad\left.\times\int_{E_s} a_{ij}(s,x)\partial_i\psi_n(x)\partial_j\psi_n(x)\rho(s,x)\,dx\,ds\right)^{\frac{1}{2}} \\
&\leqslant \mathcal{A}(\widehat{G}_1F,\widehat{G}_1F)^{\frac{1}{2}}\left(\frac{1}{2}\sum_{i,j=1}^{d}\int_{[-2n,2n]\cap T_I} h(s)\,ds\right. \\
&\qquad\left.\times\int_D\left|\partial_i\psi_n(x)\partial_j\psi_n(x)\right|\overline{\rho}(x)\,dx\right)^{\frac{1}{2}} \\
&\leqslant \mathcal{A}(\widehat{G}_1F,\widehat{G}_1F)^{\frac{1}{2}}\sqrt{2d}\left(\int_{[-2n,2n]\cap T_I} h(s)\,ds\,A^{\overline{\rho}}(\psi_n,\psi_n)\right)^{\frac{1}{2}}
\end{aligned}
$$

since $\sum_{i,j=1}^{d}\left|\partial_i\psi_n(x)\partial_j\psi_n(x)\right|\leqslant 2d\sum_{i=1}^{d}\partial_i\psi_n(x)\partial_i\psi_n(x)$. Therefore the space derivative part of $\mathcal{E}$ converges to 0 as $n\to\infty$ and the assertion follows. □

3. Examples

Let $H^{1,p}(\mathbb{R}^n)$, $p,n\geqslant 1$ be the Sobolev space of order one in $L^p(\mathbb{R}^n,dx)$, and $|\phi|_{H^{1,p}(\mathbb{R}^d)}=(\int_{\mathbb{R}^d}\sum_{i=1}^{d}|\partial_i\phi|^p+|\phi|^p dx)^{\frac{1}{p}}$. Let $H^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ denote the space of all ϕ such that $\phi\in H^{1,p}(U)$ for any $U\subset\mathbb{R}^n$, U relatively compact.

3.1. Skew Bessel processes w.r.t. a continuous function of bounded variation

(a) Skew Brownian motion w.r.t. a monotonic function

Let $\gamma : \mathbb{R}^+ \to \mathbb{R}$ be monotonely decreasing, i.e. $\gamma(s) \geqslant \gamma(t)$ for any $s \leqslant t$, and $d = 1$. In order to simplify things we let γ be continuous. This example concerns the case where $E = \mathbb{R}^+ \times \mathbb{R}$, $c \equiv 0$, $\kappa(s,t) = s + t$, $a_{ij} \equiv \delta_{ij}$. For $0 < \alpha < \beta$ let

$$\rho(s,x) := \alpha 1_{(-\infty,\gamma(s))}(x) + \beta 1_{[\gamma(s),\infty)}(x).$$

Note that $\rho(s,\cdot) \leqslant \rho(t,\cdot)$, if $s \leqslant t$. It is clear that $(\mathcal{A}, C_0^1(E))$ is closable on $L^2(\mathbb{R}^+ \times \mathbb{R}, \rho dy)$, κ, satisfies (A2), and furthermore, since $\forall G \in C_0^1(E)^+$, $t \geqslant 0$,

$$\int_E G(\kappa(s,t),x)\rho(s,x)\,dx\,ds = \int_E G(s,x)\rho(s-t,x)\,dx\,ds \leqslant \int_E G(s,x)\rho(s,x)\,dx\,ds,$$

(A3) holds. Similarly (A4) holds. Since $(A^1, D(A^1))$ (see right before Lemma 2.4 for the definition) is a recurrent Dirichlet form on $L^2(\mathbb{R}, dx)$ it follows from Lemma 2.4 that the resolvent associated with $\mathcal{E}$ is Markovian. This together with Theorems 1.8, 1.9, delivers a conservative diffusion associated to

$$\begin{aligned}\mathcal{E}(F,G) = &\int_0^\infty \left(\frac{\alpha}{2} \int_{-\infty}^{\gamma(s)} \partial_x F\, \partial_x G\, dx + \frac{\beta}{2} \int_{\gamma(s)}^{\infty} \partial_x F\, \partial_x G\, dx \right) ds \\ &- \int_E \partial_t F\, G\, \rho\, dy, \end{aligned} \tag{22}$$

where $F \in C_0^1(E)$, $G \in \mathcal{V}$. In order to identify the drift part of this diffusion we might proceed as follows. Denote by δ_x the Dirac measure in $x \in \mathbb{R}$. Integrating by parts in (22) we obtain that the generator of the diffusion is given informally by $LF = \frac{1}{2}\partial_{xx}F + \partial_t F + \frac{\beta-\alpha}{2}\partial_x F \delta_{\gamma(s)}(dx)ds$. If we can show that $\delta_{\gamma(s)}(dx)ds$ is a smooth measure then there is a unique PCAF representing this measure by Theorem 2.2. Theorem 2.3 then allows to identify the drift part. The identification (for the martingale part as well) will be done rigorously in a much more general case in the following subsection b). So we content ourselves with the following:

Lemma 3.1. *For any $N \in \mathbb{N}$ we have*

$$\int_0^N \int_{\mathbb{R}} |F|(s,x)\delta_{\gamma(s)}(dx)\,ds \leqslant \sqrt{\alpha^{-1} 16N\,\{\gamma(0) + 1 - \gamma(N)\}}\, |F|_{\mathcal{V}}$$

for every $F \in C_0^1(E)$.

Proof. We fix $N \in \mathbb{N}$. Let $F \in C_0^1(E)$, $\psi \in C_0^\infty(\mathbb{R})$, $0 \leqslant \psi \leqslant 1$, $|\partial_x \psi|_\infty \leqslant 2$, $\psi = 1$ on $[\gamma(N), \gamma(0)]$, $\psi = 0$ on $[\gamma(0)+1, \infty]$. For $s \in [0, N]$ we have

$$F(s, \gamma(s)) = -\int_{\gamma(s)}^{\gamma(0)+1} \partial_x(\psi(x) F(s, x))\, dx$$

and thus

$$\begin{aligned}
&\int_0^N \int_{\mathbb{R}} |F|(s, x) \delta_{\gamma(s)}(dx)\, ds \\
&\quad = \int_0^N \left| \int_{\gamma(s)}^{\gamma(0)+1} \partial_x(\psi(x) F(s, x))\, dx \right| ds \\
&\quad \leqslant \sqrt{\gamma(0) + 1 - \gamma(N)} \int_0^N \left(\int_{\gamma(s)}^{\gamma(0)+1} \partial_x(\psi(x) F(s, x))^2\, dx \right)^{\frac{1}{2}} ds \\
&\quad \leqslant \sqrt{8N\, \{\gamma(0) + 1 - \gamma(N)\}} \left(\int_0^N \int_{\gamma(s)}^{\gamma(0)+1} (\partial_x F(s, x))^2 + F(s, x)^2\, dx\, ds \right)^{\frac{1}{2}} \\
&\quad \leqslant \sqrt{\beta^{-1} 8N\, \{\gamma(0) + 1 - \gamma(N)\}} \left(\int_E \left((\partial_x F)^2 + F^2 \right) \rho\, dy \right)^{\frac{1}{2}}. \qquad \square
\end{aligned}$$

(b) Skew Bessel processes of dimension $\delta > 0$ w.r.t. a monotonic function

Let $\delta > 0$ be arbitrary. Let $\gamma : \mathbb{R}^+ \to \mathbb{R}$ be monotonely decreasing, i.e. $\gamma(s) \geqslant \gamma(t)$ for any $s \leqslant t$. Again, in order to simplify things we let γ be continuous. We let $c \equiv 1$, $\kappa((s, x), t) = s + t$, $a_{ij} \equiv \delta_{ij}$. For $\beta > 0$, $0 \leqslant \alpha \leqslant \beta$ let

$$\rho(s, x) := \left(\alpha 1_{(-\infty, \gamma(s))}(x) + \beta 1_{[\gamma(s), \infty)}(x) \right) |x|^{\delta - 1}.$$

Let $E = \{(s, x) \in \mathbb{R}^+ \times \mathbb{R} | x \geqslant \gamma(s)\}$ if $\alpha = 0$, $E = \mathbb{R}^+ \times \mathbb{R}$ otherwise. Note that $\rho(s, \cdot) \leqslant \rho(t, \cdot)$, and that $E_s \subset E_t$, if $s \leqslant t$. By Lemma 1.1(i) $(\mathcal{A}, C_0^1(E))$ is closable in $L^2(E, \rho dy)$. κ satisfies (A2). Furthermore, since $\forall G \in C_0^1(E)^+$, $t \geqslant 0$,

$$\begin{aligned}
\int_E G(\kappa((s, x), t), x) \rho(s, x)\, dx\, ds &= \int_E G(s, x) \rho(s - t, x)\, dx\, ds \\
&\leqslant \int_E G(s, x) \rho(s, x)\, dx\, ds,
\end{aligned}$$

(A3) holds. Similarly (A4) holds. Hence by Theorems 1.8, 1.9, for any $\delta > 0$ we obtain a diffusion up to ζ associated to

$$\mathcal{E}(F,G)=\int_0^\infty \left(\frac{\alpha}{2}\int_{-\infty}^{\gamma(s)} \partial_x F\, \partial_x G\, |x|^{\delta-1}\, dx + \frac{\beta}{2}\int_{\gamma(s)}^{\infty} \partial_x F\, \partial_x G\, |x|^{\delta-1} dx\right) ds$$
$$- \int_E \partial_t F\, G\, \rho\, dy, \tag{23}$$

where $F \in C_0^1(E)$, $G \in \mathcal{V}$. Since the closure of

$$\frac{1}{2}\int_{\mathbb{R}} \partial_x \phi\, \partial_x \psi\, |x|^{\delta-1}\, dx; \quad \phi, \psi \in C_0^\infty(\mathbb{R})$$

in $L^2(\mathbb{R}, |x|^{\delta-1}dx)$ is a recurrent Dirichlet form for $\delta \in (0,2)$ (see e.g. [6, Theorem 1.6.7 (i)]) it follows from Lemma 2.4 that the resolvent associated with $\mathcal{E}$ is Markovian for $\delta \in (0,2)$.

We will identify the corresponding diffusion when $\delta \in [1,2)$. Let $(K_n)_{n \geqslant 1}$ be an increasing sequence of compact subsets of $\mathbb{R}^+ \times \mathbb{R}$ with $\mathbb{R}^+ \times \mathbb{R} = \bigcup_{n \geqslant 1} K_n$. Let $\overline{E}_n := K_n \cap E$, $n \geqslant 1$. Since $C_0^1(E) \subset \mathcal{F}$ dense, it follows from [18, III.Remark 2.11] that $(\overline{E}_n)_{n \geqslant 1}$ is an $\mathcal{E}$-nest in the sense of [18, III.Definition 2.3(i)]. Consequently, $P_y(\lim_{n\to\infty} \sigma_{\overline{E}_n^c} < \infty) = 0$ for $\mathcal{E}$-q.e. $y \in E$, hence in particular for ρdy-a.e. $y \in E$ (see [18, IV. Lemma 3.10]). We obtain that $(\overline{E}_n)_{n \geqslant 1}$ is an strict $\mathcal{E}$-nest by (8). [22, Lemma 0.8(ii)] now implies $P_y(\lim_{n\to\infty} \sigma_{\overline{E}_n^c} < \infty) = 0$ for strictly $\mathcal{E}$-q.e. $y \in E$. We may without loss of generality assume that $\overline{E}_n \subset [0,n] \times \mathbb{R} \cap E$, $n \geqslant 1$, and that $\overline{E}_n$ is contained in the interior of $\overline{E}_{n+1}$ for any $n \geqslant 1$. From now on we will fix such a strict $\mathcal{E}$-nest $(\overline{E}_n)_{n \geqslant 1}$.

Lemma 3.2. *Let $\delta \geqslant 1$. The measure $1_{\overline{E}_N}(s,x)|x|^{\delta-1}\delta_{\gamma(s)}(dx)\, ds$, $N \geqslant 1$, is smooth w.r.t. $(\mathcal{A}, \mathcal{V})$.*

Proof. We only treat the case $\delta > 1$. The case $\delta = 1$ can be treated as in the proof of Lemma 3.1. Let $N \geqslant 1$. Let $F \in C_0^1(E)$, $\psi \in C_0^\infty(\mathbb{R})$, $0 \leqslant \psi \leqslant 1$, $|\partial_x \psi|_\infty \leqslant 2$, $\psi = 1$ on $[\gamma(N), \gamma(0)]$, $\psi = 0$ on $[\gamma(0)+1, \infty)$. For $s \in [0,N]$ we have

$$F(s,\gamma(s))|\gamma(s)|^{\delta-1} = -\int_{\gamma(s)}^{\gamma(0)+1} \partial_x \left(\psi(x) F(s,x)|x|^{\delta-1}\right) dx$$

and thus

$$\int_{\overline{E}_N} |F|(s,x)|x|^{\delta-1}\delta_{\gamma(s)}(dx)\, ds$$

$$
\begin{aligned}
&\leqslant \int_0^N \left| \int_{\gamma(s)}^{\gamma(0)+1} \partial_x \left(\psi(x) F(s,x) |x|^{\delta-1} \right) dx \right| ds \\
&\leqslant 2 \int_0^N \int_{\gamma(s)}^{\gamma(0)+1} \left(|\partial_x F| + |F| \right) |x|^{\delta-1} \, dx \, ds + (\delta - 1) \int_0^N \int_{\gamma(s)}^{\gamma(0)+1} |F| |x|^{\delta-2} \, dx \, ds \\
&\leqslant C_N \sqrt{\mathcal{A}_1(F,F)} + (\delta - 1) \int_0^N \int_{\gamma(s)}^{\gamma(0)+1} |F| |x|^{\delta-2} \, dx \, ds, \qquad (24)
\end{aligned}
$$

with $C_N = 2\sqrt{2 \int_0^N \int_{\gamma(s)}^{\gamma(0)+1} |x|^{\delta-1} dx ds}$.

Let $K \subset E$ be compact, and $\mathrm{Cap}^{\mathcal{A}}(K) = 0$. By (6)

$$
\mathrm{Cap}^{\mathcal{A}}(K) = \inf\{\mathcal{A}_1(F,F);\, F \in C^1_{0,K}(E)\},
$$

where $C^1_{0,K}(E) = \{F \in C^1_0(E) | F(s,x) \geqslant 1, \forall (s,x) \in K\}$. Hence, there exists $(F_n)_{n\in\mathbb{N}} \subset C^1_0(E)$, $F_n(s,x) \geqslant 1$, for every $n \in \mathbb{N}$, $(s,x) \in K$, such that $|F_n|_{\mathcal{V}} \to 0$ as $n \to \infty$. Since normal contractions operate on $\mathcal{V}$ we may assume that $\sup_{n\in\mathbb{N}} \sup_{(s,x)\in K} |F_n (s,x)| \leqslant C$. Selecting a subsequence if necessary we may also assume that $\lim_{n\to\infty} |F_n| = 0$ $\rho(s,x) dx\, ds$-a.e., hence $dx\, ds$-a.e. Consequently, using Lebesgue's theorem we obtain

$$
I_n := (\delta - 1) \int_0^N \int_{\gamma(s)}^{\gamma(0)+1} |F_n| |x|^{\delta-2} \, dx \, ds \to 0 \text{ as } n \to \infty.
$$

Therefore by (24)

$$
\begin{aligned}
\int_{\overline{E}_N} 1_K(s,x) |x|^{\delta-1} \delta_{\gamma(s)}(dx) \, ds &\leqslant \lim_{n\to\infty} \int_{\overline{E}_N} |F_n|(s,x) |x|^{\delta-1} \delta_{\gamma(s)}(dx) \, ds \\
&\leqslant \lim_{n\to\infty} \{ C_N \sqrt{\mathcal{A}_1(F_n,F_n)} + I_n \} = 0.
\end{aligned}
$$

Since $1_{\overline{E}_N}(s,x)|x|^{\delta-1}\delta_{\gamma(s)}(dx)ds$, as well as $\mathrm{Cap}^{\mathcal{A}}$ are inner regular we obtain that the measure $1_{\overline{E}_N}(s,x)|x|^{\delta-1}\delta_{\gamma(s)}(dx)ds$ is smooth w.r.t. $(\mathcal{A}, \mathcal{V})$. $\square$

Let us chose $(J_M)_{M\geqslant 1}$, $(H_M)_{M\geqslant 1} \subset C^2_0(E)$, with

$$
H_M(s,x) := \begin{cases} x & \text{for } (s,x) \in \overline{E}_M \\ 0 & \text{for } (s,x) \in \overline{E}^c_{M+1}, \end{cases}
$$

$M\geqslant 1$, and

$$J_M(s,x) := \begin{cases} s & \text{for } (s,x)\in \overline{E}_M \\ 0 & \text{for } (s,x)\in \overline{E}^c_{M+1}, \end{cases}$$

$M\geqslant 1$. We call $(H_M)_{M\geqslant 1}$ (resp. $(J_M)_{M\geqslant 1}$) a localizing sequence for $H(s,x) := x$ (resp. $J(s,x) := s$). Obviously

$$A^{[H_K]}_{t\wedge\sigma_{\overline{E}^c_M}} = A^{[H_L]}_{t\wedge\sigma_{\overline{E}^c_M}} \text{ for any } K\geqslant L\geqslant M.$$

We claim that

$$M^{[H_K]}_{t\wedge\sigma_{\overline{E}^c_M}} = M^{[H_L]}_{t\wedge\sigma_{\overline{E}^c_M}} \text{ for any } K\geqslant L\geqslant M.$$

Indeed, for strictly $\mathcal{E}$-q.e. $y\in E$, and any $t\geqslant 0$,

$$\begin{aligned} E_y\left[\langle M^{[H_K-H_L]}\rangle_{t\wedge\sigma_{\overline{E}^c_M}}\right] &= E_y\left[\int_0^{t\wedge\sigma_{\overline{E}^c_M}} 1_{\overline{E}_M}(Y_s)\,d\langle M^{[H_K-H_L]}\rangle_s\right] \\ &\leqslant E_y\left[\int_0^t 1_{\overline{E}_M}(Y_s)\,d\langle M^{[H_K-H_L]}\rangle_s\right]. \end{aligned}$$

By Lemma 2.1(ii) $\mu_{\int_0^\cdot 1_{\overline{E}_M}(Y_s)d\langle M^{[H_K-H_L]}\rangle_s} = \mu_{\langle M^{[H_K-H_L]}\rangle}(\overline{E}_M) = 0$. Thus by injectivity of the Revuz-correspondence (see [21, Remark 5.2(ii)]) $E_y\left[\int_0^t 1_{\overline{E}_M}(Y_s)\,d\langle M^{[H_K-H_L]}\rangle_s\right]$ $= 0$ strictly $\mathcal{E}$-q.e. $y\in E$. Hence the same is true for $E_y\left[\langle M^{[H_K-H_L]}\rangle_{t\wedge\sigma_{\overline{E}^c_M}}\right]$. We know that $\left((M_t^{[H_K-H_L]})^2 - \langle M^{[H_K-H_L]}\rangle_t\right)_{t\geqslant 0}$ is a martingale w.r.t. P_y for strictly $\mathcal{E}$-q.e. $y\in E$. The optional sampling theorem then implies

$$E_y\left[(M^{[H_K-H_L]}_{t\wedge\sigma_{\overline{E}^c_M}})^2\right] = E_y\left[\langle M^{[H_K-H_L]}\rangle_{t\wedge\sigma_{\overline{E}^c_M}}\right] = 0$$

for strictly $\mathcal{E}$-q.e. $y\in E$ and the claim is shown. The analogous statements hold for $A^{[J_K]}$, $M^{[J_K]}$. Thus we may set

$$M_t^{[H]} := \lim_{M\to\infty} M_t^{[H_M]} \qquad\qquad N_t^{[H]} := A_t^{[H]} - M_t^{[H]},$$

and

$$M_t^{[J]} := \lim_{M\to\infty} M_t^{[J_M]} \qquad\qquad N_t^{[J]} := A_t^{[J]} - M_t^{[J]},$$

in order to obtain

$$A_t^{[H]} = M_t^{[H]} + N_t^{[H]}, \qquad A_t^{[J]} = M_t^{[J]} + N_t^{[J]}.$$

Note that $N_t^{[H]} = \lim_{M\to\infty} N_t^{[H_M]}$, $N_t^{[J]} = \lim_{M\to\infty} M_t^{[J_M]}$. We want to find the explicit expressions for $M^{[J]}$, $N^{[J]}$, $M^{[H]}$, $N^{[H]}$. Let $F \in C_0^2(E)$. Integrating by parts in (23) we obtain for any $G \in C_0^1(E)$

$$\begin{aligned}
-\mathcal{E}(F,G) = & \int_E \left(\frac{1}{2}\,\partial_{xx}F + \frac{\delta-1}{2}\,x^{-1}\partial_x F + \partial_t F\right) G\rho\,dy \\
& - \frac{\alpha}{2}\int_0^\infty \int_{-\infty}^{\gamma(s)} \partial_x\left(\partial_x F\,G|x|^{\delta-1}\right)dx\,ds \\
& - \frac{\beta}{2}\int_0^\infty \int_{\gamma(s)}^{\infty} \partial_x\left(\partial_x F\,G|x|^{\delta-1}\right)dx\,ds \\
= & \int_E \left(\frac{1}{2}\,\partial_{xx}F + \frac{\delta-1}{2}\,x^{-1}\partial_x F + \partial_t F\right) G\rho\,dy \\
& + \frac{\beta-\alpha}{2}\int_E \partial_x F G|x|^{\delta-1}\delta_{\gamma(s)}(dx)\,ds.
\end{aligned} \tag{25}$$

Obviously, (25) extends to $G \in \mathcal{V}_b$. By Lemma 3.2 the measure $1_{\overline{E}_M}(s,x)|x|^{\delta-1}\delta_{\gamma(s)}(dx)ds$, $M \geqslant 1$, is smooth w.r.t. $(\mathcal{A},\mathcal{V})$. Let ℓ_t^γ denote the unique PCAF of Y associated to $|x|^{\delta-1}\delta_{\gamma(s)}(dx)ds$ by Theorem 2.2. Then $\int_0^t G(Y_s)d\ell_s^\gamma$ is associated to $G(s,x)|x|^{\delta-1}\delta_{\gamma(s)}(dx)ds$ by Theorem 2.2 for any $G \in \mathcal{B}_b(E)$. We obtain

$$\begin{aligned}
N_t^{[F]} = & \int_0^t \left(\frac{1}{2}\,\partial_{xx}F + \frac{\delta-1}{2}\,H^{-1}\partial_x F + \partial_t F\right)(Y_s)\,ds \\
& + \frac{\beta-\alpha}{2}\int_0^t \partial_x F(Y_s)\,d\ell_s^\gamma
\end{aligned} \tag{26}$$

Indeed, if we denote the r.h.s. of (26) by A_t then in particular by (25) $-\mathcal{E}(F,\widehat{G}_1 W) = \lim_{\alpha\to\infty}\alpha^2 E_{\widehat{G}_1 W\rho dy}\left[\int_0^\infty e^{-\alpha t}A_t dt\right]$ for all $W \in \mathcal{H}_b$. Hence $N_t^{[F]} = A_t$ by Theorem 2.3. On the other hand, by Lemma 2.1(i) the Revuz measure $\mu_{\langle M^{[F]}\rangle}$ is equal to $(\partial_x F)^2 dy$. A simple calculation shows that the Revuz measure of $\int_0^t (\partial_x F)^2(Y_s)ds$ is also equal to $(\partial_x F)^2 dy$. Consequently, by Trutnau [21, Remark 5.2(ii)] $\langle M^{[F]}\rangle_t = \int_0^t (\partial_x F)^2(Y_s)ds$ and therefore

$$M_t^{[F]} = \int_0^t \partial_x F(Y_s)\,dW_s \tag{27}$$

with $((W_t)_{t\geqslant 0}, P_y, (\mathcal{F}_t)_{t\geqslant 0})$ being a Brownian motion starting at zero for strictly $\mathcal{E}$-q.e. $y\in E$. By (25), (26),

$$J(Y_t) = J(Y_0) + t.$$

We put $X_t := H(Y_t)$, $t\geqslant 0$. Applying again (25), (26), we obtain

$$X_t = X_0 + W_t + \frac{\delta-1}{2}\int_0^t \frac{ds}{X_s} + \frac{\beta-\alpha}{2}\,\ell_t^\gamma. \tag{28}$$

Since $x^{-1}\in L^1_{\mathrm{loc}}(E,\rho dy)$, if $\delta>1$, and $\frac{\delta-1}{2}\int_0^t \frac{ds}{X_s}$ disappears if $\delta=1$, we see that X_t is a continuous semimartingale.

(c) The SDE from (a) and (b), written with the corresponding symmetric local time

If the distributional derivative $\gamma'\in L^2_{\mathrm{loc}}(\mathbb{R}^+)$, we claim that ℓ_t^γ in (b) is up to a constant the symmetric local time $\widetilde{L}_t^0(X-\gamma)$ at 0 of the continuous semimartingale $X_t-\gamma(t)$. In the following we will determine this constant. Let $(F_M)_{M\geqslant 1}\subset C_0^2(E)$, with

$$F_M(s,x) := \begin{cases} 1 \ \text{for} \ (s,x)\in \overline{E}_M \\ 0 \ \text{for} \ (s,x)\in \overline{E}^c_{M+1}, \end{cases}$$

$M\geqslant 1$, and $\Gamma(s,x) := |x-\gamma(s)|$. It is easy to see that $\Gamma F_M\in\mathcal{F}$ for any $M\geqslant 1$. We will use the same localization procedure as in (b). Thus, if $M_t^{[\Gamma]} := \lim_{M\to\infty} M_t^{[\Gamma F_M]}$, and $N_t^{[\Gamma]} := \lim_{M\to\infty} N_t^{[\Gamma F_M]}$, then $A_t^{[\Gamma]} = M_t^{[\Gamma]} + N_t^{[\Gamma]}$. Let

$$sign(x) := \begin{cases} 1 \ \text{if} \ \ x>0 \\ 0 \ \text{if} \ \ x=0 \\ -1 \ \text{if} \ \ x<0. \end{cases}$$

If $G\in C_0^1(E)$ then

$$\begin{aligned} -\mathcal{E}(\Gamma F_M, G) =& \frac{1}{2}\int_E \partial_x\left(\partial_x(F_M\Gamma)|x|^{\delta-1}\right) G\left(\alpha 1_{(-\infty,\gamma(s))}(x) + \beta 1_{[\gamma(s),\infty)}(x)\right) dy \\ & - \int_0^\infty \left(\frac{\alpha}{2}\int_{-\infty}^{\gamma(s)} \partial_x\left(\partial_x(F_M\Gamma)G|x|^{\delta-1}\right) dx \right. \\ & \left. + \ \frac{\beta}{2}\int_{\gamma(s)}^\infty \partial_x\left(\partial_x(F_M\Gamma)G|x|^{\delta-1}\right) dx\right) ds + \int_E \partial_t\,(F_M\Gamma)\,G\rho\,dy \\ =& \frac{1}{2}\int_E \left(\partial_{xx}(F_M\Gamma) + \partial_x(F_M\Gamma)(\delta-1)x^{-1}\right) G\rho\,dy \end{aligned}$$

$$
\begin{aligned}
&-\int_0^\infty \left[\frac{\alpha}{2}\partial_x\left(F_M(\gamma(s)-x)\right)-\frac{\beta}{2}\,\partial_x\left(F_M(x-\gamma(s))\right)\right]\\
&\times(s,\gamma(s))G(s,\gamma(s))|\gamma(s)|^{\delta-1}ds+\int_E \partial_t\,(F_M\Gamma)\,G\rho\,dy\\
=&\frac{1}{2}\int_E\Big(\Gamma\partial_{xx}F_M+2sign(x-\gamma(s))\partial_x F_M\\
&+\Gamma\partial_x F_M(\delta-1)x^{-1}\Big)\,G\rho\,dy\\
&+\int_E \frac{\delta-1}{2}\,F_M sign(x-\gamma(s))x^{-1}G\rho\,dy\\
&+\frac{\alpha+\beta}{2}\int_E F_M G|x|^{\delta-1}\delta_{\gamma(s)}(dx)\,ds\\
&+\int_E\big(|x-\gamma(s)|\partial_t F_M-F_M sign\,(x-\gamma(s))\,\gamma'(s)\big)\,G\rho\,dy. \qquad (29)
\end{aligned}
$$

Obviously (29) extends to $G\in\mathcal{V}_b$. Thus

$$N_t^{[\Gamma]}=\frac{\delta-1}{2}\int_0^t\frac{sign(X_s-\gamma(s))}{X_s}\,ds-\int_0^t sign\,(X_s-\gamma(s))\,d\gamma(s)+\frac{\alpha+\beta}{2}\ell_t^\gamma.$$

On the other hand by Lemma 2.1(i)

$$
\begin{aligned}
\mu_{\langle M^{[\Gamma F_M]}\rangle}=&\partial_x\,(|x-\gamma(s)|F_M)^2\,\rho\,dy\\
=&\Big(\big(|x-\gamma(s)|\partial_x F_M\big)^2+2|x-\gamma(s)|\partial_x F_M sign\,(x-\gamma(s))\,F_M\Big)\,\rho\,dy\\
&+(F_M sign\,(x-\gamma(s)))^2\,\rho\,dy.
\end{aligned}
$$

We obtain $\langle M^{[\Gamma]}\rangle_t=\int_0^t sign(X_s-\gamma(s))^2ds$. Consequently

$$M_t^{[\Gamma]}=\int_0^t sign(X_s-\gamma(s))\,dW_s.$$

Note that $\int_0^t sign(X_s-\gamma(s))d\ell_s^\gamma=0$, because $sign(0)=0$. Therefore

$$|X_t-\gamma(t)|=|X_0-\gamma(0)|+\int_0^t sign(X_s-\gamma(s))\,d(X_s-\gamma(s))+\frac{\alpha+\beta}{2}\,\ell_t^\gamma$$

and $\frac{\alpha+\beta}{2}\ell_t^\gamma$ must be the symmetric local time at zero of the continuous semimartingale $X_t - \gamma(t)$, i.e.

$$\frac{\alpha+\beta}{2}\,\ell_t^\gamma = \widetilde{L}_t^0(X-\gamma) := \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(X_s-\gamma(s))\,ds$$

(see [16, VI.(1.25) Exercise]). Combining the last with (28) and noting that $\frac{\beta-\alpha}{\beta+\alpha}\in[0,1]$, we obtain for every $\eta\in[0,1]$ a solution X_t to

$$X_t = X_0 + W_t + \frac{\delta-1}{2}\int_0^t \frac{ds}{X_s} + \eta\widetilde{L}_t^0(X-\gamma). \tag{30}$$

If γ increases, similarly we obtain for every $\eta\in[-1,0]$ a solution to (30).

Remark 3.3. If $\delta=1$, and if the distributional sense derivative γ' is in $L^2_{\mathrm{loc}}(\mathbb{R}^+,dx)$, without any monotonicity assumption on γ, we can do things partially better by standard methods of stochastic calculus.

Proposition 3.4. *Let $\eta\in[-1,1]$. There is a unique strong solution to*

$$X_t = X_0 + W_t + \eta\widetilde{L}_t^0(X-\gamma). \tag{31}$$

Proof. Let $(\Omega,(\mathcal{F}_t)_{t\geqslant 0},P)$ be a filtered probability space, W_t an $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian motion. We have to show that (31) has a (pathwise) unique solution. We define $B_t := W_t+\gamma(t)-\gamma(0)$, and $dQ = e^{-\int_0^t\gamma'(s)dB_s-\frac{1}{2}\int_0^t|\gamma'(s)|^2ds}dP$ on $\mathcal{F}_t$. By Girsanov's theorem, B_t is a classical $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian motion w.r.t. Q. By Harrison and Shepp [7] there is a (pathwise unique) solution to

$$Y_t = y + B_t + \eta\widetilde{L}_t^0(Y).$$

We set $X_t = Y_t+\gamma(t)$, and find a solution to (31). Using the same argument we obtain pathwise uniqueness. □

Now let us show that there is no solution to (31) if $|\eta|>1$. If there were a solution to (31) with $|\eta|>1$, we could transform (31) into the η-skew Brownian motion with $|\eta|>1$. But this process does not exist by [7].

After having completed this article we found our argument of using the Girsanov transformation in Proposition 3.4 at least in the earlier preprint [2]. [2] treats the reflected Brownian motion case, i.e. $\eta=1$.

On the other hand, if $\delta>1$, a solution to (28), (30), seems not to be attainable through a Girsanov transformation or by direct stochastic calculus. Moreover, let us show that we even do not have uniqueness of (30) in the putative simplest case $\gamma \equiv 0$. So, (30) reads as

$$X_t = X_0 + W_t + \frac{\delta - 1}{2} \int_0^t \frac{ds}{X_s} + \eta \widetilde{L}_t^0(X). \tag{32}$$

Assume that we can show $\widetilde{L}_t^0(X) \equiv 0$, and consider the case $\delta \in (1,2)$. Then of course the classical Bessel process $Bes(\delta)$ of dimension δ is a solution to (32). A second solution is given as $-Bes(\delta)$ with Brownian motion $-W_t$. Indeed, it suffices to consider $X_0 = 0$. Let $(X_t, (\mathcal{F}_t), P)$ be a solution to (32). It remains to show $\widetilde{L}_t^0(X) \equiv 0$. Applying Itô's formula we obtain that X_t^2 is a square Bessel of dimension δ. Thus $|X_t|$ must be a Bes(δ). Let us denote its $(\mathcal{F}_t)$-Brownian motion part with $\widetilde{B}_t$. Now, let us apply the "symmetric" Tanaka formula (see [16, VI. (1.25) Exercise 2^0)]) to $|X_t|$, in order to obtain

$$d|X_t| = \int_0^t sign(X_s)\, dW_s + \frac{\delta - 1}{2} \int_0^t \frac{sign(X_s)}{X_s} ds + \widetilde{L}_t^0(X).$$

The occupation times formula implies that $\int_0^t \frac{1_{\{X_s=0\}}}{X_s} ds = 0$. Therefore

$$d|X_t| = \int_0^t sign(X_s)\, dW_s + \frac{\delta - 1}{2} \int_0^t \frac{ds}{|X_s|} + \widetilde{L}_t^0(X).$$

$\beta_t := \int_0^t sign(X_s) dW_s$ is an $(\mathcal{F}_t)$-martingale, and we have $\widetilde{B}_t = \beta_t + \widetilde{L}_t^0(X)$. Hence $(\widetilde{B}_t - \beta_t)$ is an $(\mathcal{F}_t)$-martingale starting from zero, and of bounded variation. It must therefore vanish and we obtain that $\widetilde{L}_t^0(X) \equiv 0$.

3.2. Degenerate time inhomogeneous diffusions with singular drift

Let $E = \mathbb{R}^+ \times \mathbb{R}^d$, $\rho = \varphi^2$ with φ, $\partial_i \varphi \in L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d, dy)$, $1 \leqslant i \leqslant d$. Let $A = (a_{ij})_{1 \leqslant i,j \leqslant d}$, a_{ij} be locally bounded, $\partial_j a_{ij} \in L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d, \rho dy)$, $1 \leqslant i, j \leqslant d$, and

$$\sum_{i,j=1}^d a_{ij}(y) \xi_i \xi_j \geqslant 0 \quad \forall (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d,\, y \in \mathbb{R}^+ \times \mathbb{R}^d.$$

Hence (A1′) and the closability of $(\mathcal{A}, C_0^1(\mathbb{R}^+ \times \mathbb{R}^d))$ in $L^2(\mathbb{R}^+ \times \mathbb{R}^d, \rho dy)$ are satisfied (see Lemma 1.1(ii)). Let $(\mathcal{A}, \mathcal{V})$ be the closure. The Banach space $\mathcal{V}$ is the closure of $C_0^1(\mathbb{R}^+ \times \mathbb{R}^d)$ w.r.t. to the norm $|F|_{\mathcal{V}} = \left(\int_0^\infty \int_{\mathbb{R}^d} \left(\frac{1}{2}\langle A \nabla F, \nabla F\rangle + F^2\right) \rho\, dx\, ds\right)^{\frac{1}{2}}$.

Let $c \equiv 0$, $\kappa(s,t) = s + t$. Consequently (A2) holds. If

$$\rho(s,x) \leqslant \rho(t,x) \quad \forall s \leqslant t,$$

then (A3) holds. If for some constants $M \geqslant 1$, $\omega \geqslant 0$,

$$e^{\omega s} \sum_{i,j=1}^{d} a_{ij}(s,x)\rho(s,x)\xi_i\xi_j$$
$$\leqslant M e^{\omega t} \sum_{i,j=1}^{d} a_{ij}(t,x)\rho(t,x)\xi_i\xi_j \quad \forall (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d, \ 0 \leqslant s \leqslant t,$$

then it is easy to see that (A4) is satisfied. By Theorem 1.8 there is a Hunt process $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t\geqslant 0}, (Y_t)_{t\geqslant 0}, (P_y)_{y\in\mathbb{R}^+\times\mathbb{R}^d\cup\Delta})$ with state space $\mathbb{R}^+ \times \mathbb{R}^d$, life time ζ, such that the process resolvent $R_\alpha F(y) := \int_0^\infty \int_\Omega e^{-\alpha t} F(Y_t(\omega))dP_y dt$ is a s.$\mathcal{E}$-q.c. ρdy-version of the $\mathcal{E}$-resolvent $G_\alpha F$ for any $\alpha > 0$ and any $F \in \mathcal{H}_b$. By Theorem 1.9 $\mathbb{M}$ is a diffusion. $\mathbb{M}$ is related to the generalized Dirichlet form $\mathcal{E}$ which can be written as

$$\mathcal{E}(F,G) = \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^d} \langle A\nabla F, \nabla G\rangle \rho \, dx\, ds - \int_0^\infty \int_{\mathbb{R}^d} \partial_t F\, G\rho\, dx\, ds$$

if $F \in C_0^1(\mathbb{R}^+ \times \mathbb{R}^d)$, $G \in \mathcal{V}$. The corresponding generator on $C_0^1(\mathbb{R}^+ \times \mathbb{R}^d)$ is as integration by part shows (cf. proof of Lemma 1.1(ii)) the following

$$LF = \frac{1}{2}\sum_{i,j=1}^{d} \left(a_{ij}\partial_i\partial_j F + \partial_j a_{ij}\partial_i F + a_{ij}\frac{\partial_j \rho}{\rho}\partial_i F \right) + \partial_t F.$$

Suppose that there is $h \in L^1_{\text{loc}}(\mathbb{R}^+, ds)$, $\overline{\rho} \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$, such that $(A^{\overline{\rho}}, D(A^{\overline{\rho}}))$ (for the definition see right before Lemma 2.4) is recurrent, and such that $a_{ij}\rho(s,x) \leqslant h(s)\overline{\rho}(x)\, dx\, ds$-a.e., $1 \leqslant i, j \leqslant d$. Then $\mathbb{M}$ is conservative by Lemma 2.4(ii). Criteria for the recurrence of $(A^{\overline{\rho}}, D(A^{\overline{\rho}}))$ can be found in the book [6]. In this case we can identify Y similarly as in the previous example I.b). Namely, we obtain $Y_t = (t, X_t)$ with

$$dX_t = \sqrt{A}(t,X_t)\, dW_t + \frac{1}{2}\nabla A(t,X_t)\, dt + \frac{1}{2}\rho^{-1} A(\nabla\rho)(t,X_t)\, dt,$$

$\nabla A := (\sum_{j=1}^d \partial_j a_{1j}, \ldots, \sum_{j=1}^d \partial_j a_{dj})$, W_t is a d-dimensional Brownian motion starting from zero.

In the particular case $d = 1$, $\rho(s,x) = \frac{1}{A(s,x)} e^{2\int_0^x \frac{b}{A}(s,x')dx'}$, we obtain $Y_t = (t, X_t)$ with

$$dX_t = \sqrt{A}(t,X_t)\, dW_t + b(t,X_t)\, dt.$$

In the particular case $d = 1$, $\rho \equiv 1$, the only conditions on A in order to obtain $Y_t = (t, X_t)$ with

$$dX_t = \sqrt{A}(t, X_t)\, dW_t + \frac{1}{2}\, \partial_x A(t, X_t)\, dt$$

are the following:

$$A \geqslant 0 \text{ is locally bounded}, \qquad \partial_x A \in L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}, dx\, ds),$$

$$e^{\omega s} A(s, x) \leqslant M e^{\omega t} A(t, x) \qquad 0 \leqslant s \leqslant t,\ x \in \mathbb{R}.$$

A criterion for conservativity is then e.g. given (cf. [6, Theorem 1.6.7]) if there exists some $h \in L^1_{\text{loc}}(\mathbb{R}^+, dx)$ with

$$A(s, x) \leqslant h(s)|x|^{\alpha}, \ \ \alpha < 1.$$

3.3. Non-conservative and exponential time scale diffusions

Suppose that E, A, ρ, c, satisfy the conditions of Section 1. Let $\kappa(s, t) = s e^{kt}$, $t \geqslant 0$, with positive constant k. By the results of Sections 1, 2, we obtain a strong Markov process which is a diffusion up to its life time and which has informally

$$\begin{aligned} LF(s, x) =& \sum_{i,j=1}^{d} a_{ij}(s, x)\partial_i\partial_j F(s, x) + \sum_{i=1}^{d} b_i(s, x)\partial_i F(s, x) - c(s, x)F(s, x) \\ &+ sk\partial_t F(s, x), \end{aligned}$$

with $b_i = \frac{1}{2}\sum_{j=1}^{d}\left(\partial_j a_{ij} + a_{ij}\frac{\partial_j \rho}{\rho}\right)$, $1 \leqslant i \leqslant d$, as infinitesimal generator.

If the space direction is 1-dimensional, we may write $\rho(s, x) = \frac{1}{A(s,x)} e^{2\int_0^x \frac{b}{A}(s,x')dx'}$, so that

$$LF(s, x) = \frac{A}{2}(s, x)\Delta_x F(s, x) + b(s, x)\partial_x F(s, x) - c(s, x)F(s, x) + sk\partial_t F(s, x).$$

If $c \neq 0$ the diffusion corresponding to L cannot be conservative.

On the other hand, let $c \equiv 0$ with suitable A, b, satisfying e.g. the conditions of Lemma 2.4. Then $Y_t = (Z_t, X_t)$ with

$$Z_t = Z_0 e^{kt}$$

and $P_{(s,x)}(Z_0 = s) = 1$ for strictly $\mathcal{E}$-q.e. $(s, x) \in E$.

Indeed, let $J(s,x) = s$, $H(s,x) = x$ (at least locally). Then $Z_t = J(Y_t)$, $X_t = H(Y_t)$, and

$$Z_t - Z_0 = M_t^{[J]} + N_t^{[J]}.$$

Since the energy measure $\mu_{\langle J \rangle}$ related to the quadratic variation $\langle M^{[J]} \rangle$ of the continuous martingale $M_t^{[J]}$ vanishes, we must have $M^{[J]} \equiv 0$. On the other hand

$$N_t^{[J]} = \int_0^t LJ(Y_s)\,ds = \int_0^t kJ(Y_s)\,ds = \int_0^t kZ_s\,ds.$$

Therefore

$$Z_t = Z_0 + \int_0^t kZ_s\,ds$$

and the assertion follows. Thus $Y_t = (Z_0 e^{kt}, X_t)$.

3.4. Moving Lipschitz domains

Let $E_0 \subset \mathbb{R}^d$ be closed. Let $(E_s)_{s>0}$, be a family of Euclidean closures of (not necessarily bounded) Lipschitz domains in $\mathbb{R}^d$ such that $E_s \subset E_t \subset \mathbb{R}^d$, $0 \leqslant s \leqslant t$. Let us assume that also $E = \bigcup_{t \in \mathbb{R}^+} \{t\} \times E_t$ is the Euclidean closure of a Lipschitz domain in $\mathbb{R}^{d+1}$. Then $E \setminus \partial E$ is of $(d+1)$-dimensional Lebesgue measure zero. Let $A = (a_{ij})_{1 \leqslant i,j \leqslant d}$, satisfy (A1). Let $\rho \in H^{1,1}_{\mathrm{loc}}(\mathbb{R}^{d+1})$, $\rho > 0$ *dy*-a.e. By [9, Theorem 1, p. 8] there exists a *dy*-version of ρ which is absolutely continuous on almost all straight lines which are parallel to the coordinate axes. This version restricted to an arbitrary relatively compact subset $U \subset \mathbb{R}^{d+1}$ then satisfies condition (5.7) in [1, Theorem 5.3], see also Remark 2.3(iii) of the same article. It then follows from [1, Theorem 5.3] that

$$\frac{1}{2}\int_U \langle \nabla F, \nabla G \rangle \rho\,dy, \quad F, G \in C_0^\infty(\mathbb{R}^{d+1}),$$

is closable in $L^2(U, \rho dy)$ for any $U \subset \mathbb{R}^{d+1}$, U relatively compact. From this it is easy to see that

$$\mathcal{A}(F,G) := \frac{1}{2}\int_E \langle A\nabla F, \nabla G \rangle \rho\,dy, \quad F, G \in C_0^1(E),$$

is also closable in $L^2(E, \rho dy)$. We denote the closure by $(\mathcal{A}, \mathcal{V})$ and suppose that $a_{ij} \in \mathcal{V}_b$, $1 \leqslant i, j \leqslant d$. Thus $c \equiv 0$. Obviously (A1) is satisfied and $\kappa(s,t) := s + t$, $s, t \geqslant 0$, satisfies (A2). Suppose that

$$\rho(s,\cdot) \leqslant \rho(t,\cdot) \;\; dx\text{-a.e. for any } 0 \leqslant s \leqslant t.$$

Then (A3), (A4), is easily seen to be satisfied. Let $B_r^n(z)$ denote the open ball in $\mathbb{R}^n$ with radius r, and center z, $\oint$ denote the normalized integral. Let $z \in \partial E$, and

$$\rho^*(z) \equiv \begin{cases} \lim_{r\to 0} \oint_{B_r^{d+1}(z)\cap E} \rho(y)\, dy & \text{if the limit exists} \\ 0 & \text{otherwise.} \end{cases}$$

ρ^* is called the (precise representative of the) trace of ρ on ∂E. Let $\sigma_0(dx)$ be an arbitrary finite measure on ∂E_0. Let $\sigma_s(dx)$ be the surface measure of ∂E_s, $s > 0$. Let $\sigma(dy)$ be the surface measure of ∂E. It is well known (see e.g. [5, Section 4.3, Theorem 1(i)]) that ρ^* exists $\sigma(dy)$-a.e., hence clearly also $\sigma_s(dx)ds$-a.e.

Theorem 3.5. *Let $\overline{K} \subset E$ be compact. Then $1_{\overline{K}\cap\partial E}\rho^*\sigma_s(dx)ds$, $1_{\overline{K}\cap\partial E\cap\{\rho^*>0\}}\sigma_s(dx)\, ds$, is smooth w.r.t. $(\mathcal{A}, \mathcal{V})$.*

Proof. Since $1_{\overline{K}\cap\partial E}\rho^*\sigma_s(dx)ds$, $1_{\overline{K}\cap\partial E\cap\{\rho^*>0\}}\sigma_s(dx)ds$, are equivalent finite measures it is enough to show that $1_{\overline{K}\cap\partial E}\rho^*\sigma_s(dx)ds$ is smooth w.r.t. $(\mathcal{A}, \mathcal{V})$. Since $\overline{K}$ is compact there exists $T, r, r' > 0$, with $B_{r'}^{d+1}(0) \supset \bigcup_{t\in[0,T]} \{t\}\times(E_t \cap B_r^d(0)) \supset \overline{K}$. Let $(\rho_k)_{k\in\mathbb{N}} \subset C^\infty\left(\overline{B_{r'}^{d+1}(0) \cap (E \setminus \partial E)}\right)$ such that $\rho_k \to \rho$ in $H^{1,1}(B_{r'}^{d+1}(0) \cap (E \setminus \partial E))$ as $k \to \infty$. Then $\rho_k(y) \to \rho^*(y)$ for $\sigma(dy)$-a.e. $y \in \partial E \cap B_{r'}^{d+1}(0)$, hence also for $\sigma_s(dx)ds$-a.e. $y \in \partial E \cap B_{r'}^{d+1}(0)$, as $k \to \infty$. Let $K \subset E$ be compact, and $F \in C_0^1(E)$, $F \geqslant 1$ everywhere on K. We may assume that $(E \setminus \partial E) \cap B_{r'}^{d+1}(0)$ is connected. Then (cf. e.g. [5, Section 4.3, $(\star\star\star)$]) for some universal constant C' depending only on the Lipschitz domain $(E \setminus \partial E) \cap B_{r'}^{d+1}(0)$

$$\begin{aligned}
&1_{\overline{K}\cap\partial E}\rho^*\sigma_s(dx)\, ds(K) \\
&\quad\leqslant \liminf_{k\to\infty} \int_0^T \int_{\partial E_s \cap B_r^d(0)} |F\rho_k| \sigma_s(dx)\, ds \\
&\quad\leqslant C' \liminf_{k\to\infty} \int_0^T \int_{E_s\cap B_r^d(0)} (|\nabla(F\rho_k)| + |F\rho_k|)\, dx\, ds \\
&\quad= C' \int_0^T \int_{E_s\cap B_r^d(0)} (|\nabla F| + |F|)\rho\, dx\, ds + \int_0^T \int_{E_s\cap B_r^d(0)} |\nabla\rho||F|\, dx\, ds \\
&\quad\leqslant C'\sqrt{2(C\vee 1)} \left(\int_0^T \int_{E_s\cap B_r^d(0)} \rho\, dy\right)^{\frac{1}{2}} \mathcal{A}_1(F, F)^{\frac{1}{2}} \\
&\qquad + \int_{B_{r'}^{d+1}(0)\cap E} |\nabla\rho||F|\, dy.
\end{aligned} \tag{33}$$

Suppose that $\mathrm{Cap}^{\mathcal{A}}(K)=0$. By (6) there exist $(F_n)_{n\in\mathbb{N}}\subset C_0^1(E)$, $F_n\geqslant 1$ everywhere on K, for every $n\in\mathbb{N}$, with $\lim_{n\to\infty}\mathcal{A}_1(F_n,F_n)^{\frac{1}{2}}=0$. Since normal contractions operate on $\mathcal{V}$ we may assume that $\sup_{n\in\mathbb{N}}\sup_{y\in E}|F_n(y)|\leqslant const$. We may further assume that $\lim_{n\to\infty}|F_n|=0$ $\rho(y)dy$-a.e., hence dy-a.e. Consequently, by Lebesgue's theorem

$$\lim_{n\to\infty}\int_{B_{r'}^{d+1}(0)\cap E}|\nabla\rho||F_n|\,dy=0$$

and therefore $1_{\overline{K}\cap\partial E}\rho^*\sigma_s(dx)\,ds(K)=0$ by (33). Since $1_{\overline{K}\cap\partial E}\rho^*\sigma_s(dx)ds$ as well as $\mathrm{Cap}^{\mathcal{A}}$ are inner regular, the assertion follows. $\square$

Let $\widetilde{a}_{ij}$, $1\leqslant i,j\leqslant d$, denote $\mathcal{A}$-q.c. ρdy-versions of a_{ij}, and let $\widetilde{A}=(\widetilde{a}_{ij})_{1\leqslant i,j\leqslant d}$. Let $\eta_s=(\eta_s^1,\dots,\eta_s^d)$ be the inward normal of E_s. Let $F,G\in C_0^1(E)$. We call

$$\widetilde{A}(\eta_\cdot)(s,x)=\left(\sum_{j=1}^d\widetilde{a}_{1j}(s,x)\eta_s^j(x),\dots,\sum_{j=1}^d\widetilde{a}_{dj}(s,x)\eta_s^j(x)\right)$$

the conormal direction associated with A. Using suitable approximations for $\widetilde{a}_{ij}$, ρ, [5, Section 4.3, Theorem 1(i), p. 133], Theorem 3.5, an integration by parts similar as in the proof of Lemma 1.1(ii) gives

$$\begin{aligned}\mathcal{A}(F,G)=&\int_0^\infty\int_{E_s}-\frac{1}{2}\sum_{i,j=1}^d\left(a_{ij}\partial_i\partial_jF+\partial_ja_{ij}\partial_iF+a_{ij}\,\frac{\partial_j\rho}{\rho}\,\partial_iF\right)G\rho\,dx\,ds\\&+\frac{1}{2}\int_0^\infty\int_{\partial E_s}G(s,x)\langle\nabla F,\widetilde{A}(\eta_\cdot)\rangle(s,x)\rho^*(s,x)\sigma_s(dx)\,ds,\end{aligned}\qquad(34)$$

where $\langle\nabla F,\widetilde{A}(\eta_\cdot)\rangle(s,x)=\sum_{i,j=1}^d\partial_iF(s,x)\widetilde{a}_{ij}(s,x)\eta_s^j(x)$.

In order to simplify things let us assume that the resolvent $(G_\alpha)_{\alpha>0}$ associated with $\mathcal{E}$ is Markovian. By Theorems 1.8, 1.9, there is a conservative diffusion $\mathbb{M}=(\Omega,(\mathcal{F}_t)_{t\geqslant0},(Y_t)_{t\geqslant0},(P_y)_{y\in E\cup\Delta})$ with state space E, such that the process resolvent $R_\alpha F(y):=\int_0^\infty\int_\Omega e^{-\alpha t}F(Y_t(\omega))dP_ydt$ is a s. $\mathcal{E}$-q.c. ρdy-version of the $\mathcal{E}$-resolvent $G_\alpha F$ for any $\alpha>0$ and any $F\in\mathcal{H}_b$. By Theorem 3.5, Theorem 2.2, there exist uniquely determined PCAF's ℓ^{ρ^*}, ℓ, associated to $\rho^*\sigma_s(dx)ds$, $1_{\{\rho^*>0\}}\sigma_s(dx)ds$. Using (34) a localization procedure as in example I.b) allows to identify Y_t as (t,X_t) with

$$dX_t=\sqrt{A}(t,X_t)\,dW_t+\frac{1}{2}\left(\nabla A+\rho^{-1}A(\nabla\rho)\right)(t,X_t)\,dt+\frac{1}{2}\,\widetilde{A}(\eta_\cdot)\rho^*(t,X_t)\,d\ell_t$$

where $\nabla A := (\sum_{j=1}^{d} \partial_j a_{1j}, \ldots, \sum_{j=1}^{d} \partial_j a_{dj})$, and W_t is a d-dimensional Brownian motion starting from zero.

Applying Theorem 2.2 we can see that $\ell_t^{\rho^*} = \int_0^t \rho^*(s, X_s) d\ell_s$, $\ell_t = \int_0^t 1_{\partial E \cap \{\rho^* > 0\}}(s, X_s) d\ell_s$. Thus $\int_0^t \widetilde{A}(\eta_.)\rho^*(s, X_s) d\ell_s$ increases only when $Y_t = (t, X_t)$ meets the boundary at those points where $\rho^* > 0$. In this case Y_t is instantaneously reflected in the conormal direction $\widetilde{A}(\eta_.)$ associated with A.

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